Lodz University of Technology

Doctoral Thesis

Dynamics of thin functionally graded cylindrical shells - tolerance modelling

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List of symbols

$x \equiv x^1$, $\xi \equiv x^2$ coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively; at the same time $x \equiv x^1$ and $\xi \equiv x^2$ are points in $\Omega = (0, L_1) \in E^1$ and $\Xi \equiv (0, L_2) \in E^1$, respectively, where $L \equiv L_1$, L_2 are the length dimensions of M along x-coordinate and ξ-coordinate, respectively; it means that $(\Omega \times \Xi) \in E^2$ is referred as a region of the shell midsurface parameters x^1, x^2;

α, β indices taking values $1, 2$ and related to midsurface parameters x^1, x^2, summation convention holds;

a, b non-tensorial indices, run over $\{1, 2, \ldots, n\}$, summation convention holds;

A, B non-tensorial indices, run over $\{1, 2, \ldots, N\}$, summation convention holds;

$a_{\alpha\beta}$ covariant midsurface first metric tensor, for orthonormal parametrization introduced on M tensors $a_{\alpha\beta} = a^{\alpha\beta}$ are the unit tensors;

$b_{\alpha\beta}$ covariant midsurface second metric tensor, for orthonormal parametrization introduced on M $b_{22} = b_{12} = b_{21} = 0$ and $b_{11} = -r^{-1}$, where r is a midsurface curvature radius;

$\partial_\alpha = \partial/\partial x^\alpha$ partial differentiation with respect to x^α, $\partial_{\alpha,\ldots,\delta} \equiv \partial_\alpha \ldots \partial_\delta$;

$\Delta \equiv [-\lambda/2, \lambda/2]$ basic cell in $\Omega \in E^1$, where $\lambda \equiv \lambda_1$ is a cell length dimension in $x \equiv x^1$-direction;

$\Delta(x) \equiv x + \Delta = [-\lambda/2, \lambda/2]$ an arbitrary cell with a centre at point $x \in \Omega_\Delta$, $\Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\}$, Ω_Δ is a set of all the cell centres which are inside Ω;

λ diameter of a closed subset $\Delta \equiv [-\lambda_1/2, \lambda_1/2] \times \ldots \times [-\lambda_m/2, \lambda_m/2]$ of E^m, called the microstructure length parameter; if $m = 1$ then $\Delta \equiv [-\lambda/2, \lambda/2]$ and $\lambda \equiv \lambda_1$ is a cell length dimension in $x \equiv x^1$-direction;

$\Delta_\varepsilon \equiv (-\varepsilon\lambda/2, \varepsilon\lambda/2)$ scaled cell in $\Omega \in E^1$, $\varepsilon \in (0, 1]$;

$d(x)$	the shell thickness, $x \in \Omega$;
r	midsurface curvature radius;
t	time coordinate, $t \in \mathrm{I} \equiv [t_0, t_1]$;
$u_\alpha(x, \xi, t)$	displacements along x^α, $(x \equiv x^1, \xi \equiv x^2, t) \in \Omega \times \Xi \times \mathrm{I}$;
$w(x, \xi, t)$	displacement in the direction normal to the shell midsurface M, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$;
$D^{\alpha\beta\gamma\delta}(x)$	membrane stiffness tensor, $x \in \Omega$;
$B^{\alpha\beta\gamma\delta}(x)$	bending stiffness tensor, $x \in \Omega$;
$\mu(x)$	mass density per midsurface unit area, $x \in \Omega$;
$f^a(x, \xi, t)$	external forces along x^α, $(x \equiv x^1, \xi \equiv x^2, t) \in \Omega \times \Xi \times \mathrm{I}$;
$f(x, \xi, t)$	external forces in the direction normal to the shell midsurface M, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$;
$A(u_\alpha, w)$	action functional;
$L(\cdot)$	Lagrange function for the considered problem;
δ	set of tolerance parameters, $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$, where λ is related to the distances between points in region $\Omega \in E^m$ and δ_0, δ_k, $k = 1, 2, \ldots, R$, are related to the differences between the values of function $F(\cdot)$ defined in $\Omega \in E^m$ and its gradient $\partial^k F(\cdot)$ in points $\mathbf{x}, \mathbf{y}$ belonging to $\Omega \in E^m$ such that $\lvert \mathbf{x} - \mathbf{y} \rvert \leq \lambda$; nonnegative integer R is assumed to be specified in every problem under consideration; in the present dissertation $m = 1$;
$O(\delta)$	terms of the order of tolerance parameters δ;
$\langle f \rangle (\cdot) = \left\langle \widetilde{f} \right\rangle (\cdot)$	averaging of function f in $\Delta(\cdot)$, where $\widetilde{f}$ is a periodic approximation of f in $\Delta(\cdot)$;
$TP^R_\delta(\Omega, \Delta)$	system of tolerance-periodic functions of the *R-th* kind defined on $\Omega \in E^m$, which are tolerance-periodic with respect to cell Δ and tolerance parameters δ; in the present dissertation $m = 1$ and R is equal to either 0 or 1 or 2;
$SV^R_\delta(\Omega, \Delta)$	system of slowly-varying functions of the *R-th* kind defined on $\Omega \in E^m$, which are slowly-varying with respect to cell Δ and tolerance parameters δ; in the present dissertation $m = 1$ and R is equal to either 1 or 2;
$FS^R_\delta(\Omega, \Delta)$	system of tolerance-periodic fluctuation shape functions of the *R-th* kind defined on $\Omega \in E^m$, which are tolerance-periodic with respect to cell Δ and tolerance parameters δ; in the present dissertation $m = 1$ and R is equal to either 1 or 2;

$u^0_\alpha(\cdot,\xi,t)\in SV^1_\delta(\Omega,\Delta)$, $w^0(\cdot,\xi,t)\in SV^2_\delta(\Omega,\Delta)$	macrodisplacements (averaged variables) being unknowns of the tolerance model equations, $(\xi,t)\in\Xi\times\mathrm{I}$;
$U^a_\alpha(\cdot,\xi,t)\in SV^1_\delta(\Omega,\Delta)$, $W^A(\cdot,\xi,t)\in SV^2_\delta(\Omega,\Delta)$	fluctuation amplitudes being unknowns of the tolerance model equations, $(\xi,t)\in\Xi\times\mathrm{I}$;
$h^a(\cdot)\in FS^1_\delta(\Omega,\Delta)$, $g^A(\cdot)\in FS^2_\delta(\Omega,\Delta)$	fluctuation shape functions;
$\langle L_{hg}\rangle(x)$	the tolerance averaging of lagrangian L in $\Delta(x)$, $x\in\Omega_\Delta$;
$A_{hg}(u^0_\alpha,U^a_\alpha,w^0,W^A)$	tolerance averaging of action functional $A(u_\alpha,w)$;
$u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$	unknowns of the consistent asymptotic model, which as in the tolerance modelling are called macrodisplacements, but they are not referred to the slowly-varying functions introduced in the tolerance averaging, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$;
$U^a_\alpha(x,\xi,t)$, $W^A(x,\xi,t)$	unknowns of the consistent asymptotic model, which as in the tolerance modelling are called fluctuation amplitudes, but they are not referred to the slowly-varying functions introduced in the tolerance averaging, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$;
$L_0(\cdot)$	averaged form of lagrangian under consistent asymptotic averaging;
$u_{0\alpha}(x,\xi,t)$, $w_0(x,\xi,t)$	the known solutions to a certain initial-boundary value problem for the consistent asymptotic equations derived in the first step of the combined asymptotic-tolerance modelling, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$;
$Q^k_\alpha(\cdot,\xi,t)\in SV^1_\delta(\Omega,\Delta)$, $V^K(\cdot,\xi,t)\in SV^2_\delta(\Omega,\Delta)$	fluctuation amplitudes being unknowns of the tolerance model equations derived in the second step of the combined asymptotic-tolerance modelling, $(\xi,t)\in\Xi\times\mathrm{I}$, $k=1,2,\ldots,m$, $K=1,2,\ldots,M$;

ω_{-}, ω_{+}	fundamental lower ω_{-} and new additional cell-dependent higher ω_{+} free vibration frequencies derived from the tolerance model of a functionally graded shell strip, cf. Subsection 6.2;
ω^{AM}	free vibration frequency obtained from the asymptotic model of a functionally graded shell strip, cf. Subsection 6.2;
$\widehat{\omega}_{mn-}$, $\widehat{\omega}_{mn+}$	fundamental lower $\widehat{\omega}_{mn-}$ and new additional cell-dependent higher $\widehat{\omega}_{mn+}$ free vibration frequencies derived from the tolerance model of a functionally graded open shell of finite all length dimensions, cf. Subsection 6.3;
$\widehat{\omega}_{mn}^{AM}$	free vibration frequency obtained from the asymptotic model of a functionally graded open shell of finite all length dimensions, cf. Subsection 6.3;
$\overline{\omega}$, $\breve{\omega}$, ω	cell-dependent higher free vibration frequencies in circumferential and axial directions as well as in direction normal to the shell midsurface, respectively, studied on the basis on superimposed microscopic model equations derived in the second step of the combined asymptotic-tolerance modelling (equations independent of solutions obtained in the framework of asymptotic model formulated in the first step of the combined modelling), cf. Subsection 7.2;
c	new wave propagation speed depending on λ.

1. Introduction

1.1. Subject-matter of the doctoral thesis

The objects of considerations are thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells having a functionally graded macrostructure and a tolerance-periodic microstructure in circumferential direction.

It means, that ***on the microscopic level***, the shells under consideration consist of a very large number of separated, small elements regularly distributed along circumferential direction and perfectly bonded to each other (or to the homogeneous matrix). These elements, called ***cells***, are treated as thin shells. It is assumed that ***the adjacent cells are nearly identical*** (*i.e. they have nearly the same geometrical, elastic and inertial properties*), ***but the distant elements can be very different.*** The length dimension of a cell in circumferential direction, called ***the microstructure length parameter***, is assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the length dimension of the shell midsurface in the direction of tolerant periodicity. Examples of such shells are shown in Figs. 4.1 and 4.2. At the same time, the shells have constant structure in axial direction. ***On the microscopic level***, the geometrical, elastic and inertial properties of these shells are determined by highly oscillating, non-continuous and *tolerance-periodic functions* in circumferential direction. Roughly speaking, by *tolerance-periodic functions* we shall mean functions which in every cell can be approximated by periodic functions.

On the other hand, ***on the macroscopic level***, the averaged properties of the shells are described by functions being ***continuous and slowly varying*** along circumferential direction. It means that the tolerance-periodic shells under consideration can be treated as made of ***functionally graded materials*** (FGM), cf. Suresh and Mortensen [110], and called ***functionally graded shells.*** Moreover, since macroscopic properties of the shells are graded in direction normal to interfaces between constituents, this gradation is referred to as ***the transversal gradation.***

The subject-matter of this doctoral thesis is the analytical modelling of dynamic problems for the shells under consideration and the investigation of the effect of a cell size on the macroscopic and microscopic shell behaviour (*the length-scale effect*).

The considerations will be based on the well-known Kirchhoff-Love theory of thin linearly elastic cylindrical shells in which terms depending on the second metric tensor of the shell midsurface are neglected in the formulae for curvature changes, cf. Kirchhoff [49], Love [60], Kaliski [46]. For periodic or tolerance-periodic shells, the exact partial differential equations of this theory include strongly oscillating, non-continuous and periodic or tolerance-periodic coefficients. That is why the direct application of these equations to investigations of specific problems is non-effective even using computational methods.

To obtain averaged equations with constant or continuous and slowly-varying coefficients, a lot of different approximate modelling methods have been proposed. Periodic and tolerance-periodic structures are usually described using *homogenized models* derived by means of *asymptotic methods*, cf. Chapter 2. These models represent certain equivalent structures with constant or slowly varying material properties. Unfortunately, *in models of this kind the effect of the microstructure size on the overall shell behaviour is neglected in the first approximation which is usually employed.*

An alternative (i.e. non-asymptotic) approach to the modelling of micro-heterogeneous media was proposed by Woźniak in a series of papers and summarized in monographs by Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164]. This technique is called ***the tolerance modelling method.*** The concept of ***tolerance relations*** between points and real numbers related to the accuracy of the performed measurements and calculations plays a leading role in formulation of this technique. The tolerance relations are determined by ***the tolerance parameters.*** The second basic concepts of this method is ***a function slowly-varying within a cell.*** It is a function which, together with its derivatives occurring in the problem under consideration, can be treated as constant within every cell. The basic assumptions of this modelling technique are called ***the micro-macro decomposition and the tolerance averaging approximation.*** The first assumption states that the displacement fields can be decomposed into macroscopic and microscopic parts. The macroscopic part is represented by ***unknown averaged slowly-varying displacements.*** The highly oscillating microscopic part is described by ***the periodic or tolerance-periodic known fluctuation shape functions*** and by ***unknown slowly-varying fluctuation amplitudes***. The second assumption states that in the course of modelling the terms of the orders of tolerance parameters are neglected. The fundamental concepts and assumptions of the tolerance modelling procedure are presented in Chapter 3. Models obtained in the framework of the tolerance modelling procedure are called ***the tolerance models. The governing equations of the these models have coefficients which are constant or continuously slowly-varying and depend on the microstructure size.*** Hence, the main advantage of the tolerance models is that in contrast to asymptotic models they make it possible to describe the effect of microstructure size on a shell behaviour not only on the micro-structural level but also on the macroscopic one.

The overview of the modelling techniques applied to micro-heterogeneous structures is given in Chapter 2.

In the presented dissertation, the tolerance averaging technique is adopted to the modelling of the known governing equations of Kirchhoff-Love theory of thin linearly elastic cylindrical shells. These equations will be taken in the form of Euler-Lagrange equations generated by the Lagrange function describing behaviour of the shells in the framework of the theory under consideration. Functional coefficients of this function are tolerance-periodic, highly oscillating and often non-continuous in circumferential direction. The tolerance averaging of the Lagrange function under *micro-macro decomposition* and under *the tolerance averaging approximation* leads to the averaged form of this function with continuous and slowly-varying coefficients depending on the microstructure size. Then, applying the principle of stationary action to *the tolerance-averaged action functional* defined by means of the averaged lagrangian, we arrive at *the governing equations of tolerance model for the shells under consideration. Coefficients of these equations are continuous and slowly-varying along arc coordinate and some of them depend on the microstructure length parameter.* It means that the resulting tolerance model equations describe *the effect of the cell size on the overall shell dynamics.*

In order to evaluate the length-scale effect in some special dynamic problems, the results obtained by applying the tolerance modelling procedure are compared with those derived from asymptotic model of the functionally graded shells under consideration. In order to formulate this model, a certain new approach to the asymptotic modelling of micro-heterogeneous media proposed in Woźniak *et al.* (eds.) [164] is adopted. This new approach, called ***the consistent asymptotic modelling***, does not take into account the influence of the microstructure size on the overall shell behaviour.

The dynamic problems for the shells under consideration are also modelled by means of combined procedure, proposed by Woźniak et al. (eds.) [164], in which tolerance non-asymptotic and consistent asymptotic techniques are combined together into a new procedure. *An important advantage of the combined model is that under special conditions imposed on the fluctuation shape functions it makes it possible to separate the macroscopic* *description of some special problems from their microscopic description.* Moreover, it will be also shown that in the framework of combined model we can analyse the near-boundary and the near-initial phenomena related to the specific form of boundary and initial conditions imposed on micro-fluctuations of displacements.

The derived averaged models will be applied to investigations of the length-scale effect in some special problems of dynamics for the transversally graded shells under consideration. Because equations of the proposed models have slowly-varying functional coefficients hence it is difficult to find exact analytical solutions to these equations. To solve the vibration or wave propagation problems discussed here the known Ritz or Galerkin approximate methods will be applied.

1.2. Aims of the doctoral thesis

Thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells with a smooth, slowly varying transversal gradation of macroscopic properties and with a tolerance-periodic microstructure in circumferential direction are analysed.

The first aim of this doctoral thesis is to formulate and discuss three new mathematical averaged models for the analysis of selected dynamic problems in the cylindrical shells under consideration:

- ***tolerance model with continuous and slowly-varying coefficients depending on a cell size***, derived by applying a certain new approach to the tolerance modelling of micro-heterogeneous solids presented in Woźniak *et al.* (eds.) [164],

- ***consistent asymptotic model with continuous and slowly-varying coefficients being independent of the microstructure size***, obtained by using a certain new approach to the asymptotic modelling of micro-heterogeneous media proposed in Woźniak *et al.* (eds.) [164],

- ***combined asymptotic-tolerance model with continuous and slowly-varying coefficients depending on the cell size***, derived by applying the combined modelling which includes both the tolerance and asymptotic procedures; this technique has been presented in Woźniak *et al.* (eds.) [164]; the main advantage of this model is that it makes it possible to study micro-dynamics of the tolerance-periodic shells independently of their macro-dynamics.

The resulting equations of the above models will be presented in the form of Euler-Lagrange equations and also in the form of the dynamic equilibrium equations together with the constitutive relations.

The second aim of the dissertation is to apply the tolerance and asymptotic models derived here to evaluation of the length-scale effect in some special problems dealing with dynamics (free vibrations) of the micro-heterogeneous shells under consideration.

The third aim is to apply the combined model to the analysis of length-scale effect in some special problems for micro-dynamics of the shells under consideration. It will be shown that the combined model makes it possible to separate the macroscopic description of some special problems from the microscopic description of these problems.

Theses of the doctoral dissertation are:

- The tolerance and the consistent asymptotic models of dynamic problems for the functionally graded shells under consideration derived here can be successfully applied to analyse the macroscopic behaviour of these shells. Moreover, the tolerance model makes it possible to determine and study some phenomena related to existence of microstructure length-scale effect, e.g. the occurrence of *the additional higher-order cell-dependent free vibration frequencies.*

- The proposed combined asymptotic-tolerance model of dynamic problems for the shells under consideration allows us to successfully *study cell-dependent micro-vibrations of the tolerance-periodic shells independently of the shells' cell-independent macro-vibrations*. Moreover, this model makes it possible to analyse *length-scale effect in wave propagation problems as well as in boundary layer phenomena related to micro-fluctuations of the shell displacements.*

1.3. Scope of the doctoral thesis

The dissertation begins with a list of symbols.

The object and aim of the doctoral dissertation are specified in **Chapter 1**.

An overview of the modelling techniques applied to periodic/tolerance-periodic structures is given in **Chapter 2**.

To make considerations self-consistent, **in the subsequent chapter** we shall outline the basic concepts and assumptions of the tolerance modelling technique and of the consistent asymptotic approach, following monographs by Woźniak *et al.* (eds.) [164] and Ostrowski [90].

The shell geometry is described and the cell is defined in **Chapter 4**. In this chapter there are also shown the governing equations of the Kirchhoff-Love theory of thin elastic cylindrical shells being a starting point of the modelling procedures.

In **Chapter 5**, the tolerance, asymptotic and combined models for the analysis of special dynamic problems in functionally graded shells under consideration are derived and discussed in detail.

In **Chapter 6**, there are shown applications of the proposed tolerance and asymptotic models to analysis of the length-scale effect in some special problems for dynamics of the shells under consideration. The comparison and discussion of the results are presented. Moreover, some results derived in the framework of the tolerance and asymptotic models are compared with those obtained from the commercial software Ansys based on the finite element method.

In **Chapter 7**, there are shown applications of the proposed combined asymptotic-tolerance model to investigations of selected problems of the shell micro-dynamics as cell-dependent free micro-vibrations, the long wave propagation problem related to micro-fluctuations and certain boundary-layer phenomena.

Final remarks and conclusions are formulated in **the last chapter**. This chapter ends with a list of the most important original elements of the doctoral dissertation and the anticipated directions of further research.

The doctoral thesis is finished by Appendix, the list of references, summary, and summary in Polish. Appendix deals with calculations of coefficients in averaged models equations describing the dynamic problems discussed in the application part of this dissertation, i.e. in Chapters 6, 7.

The functionally graded shells being objects of considerations in this doctoral dissertation are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as elements of housings of reactors and tanks. Micro-heterogeneous shells having small length dimensions are elements of air-planes, ships and machines.

Note, that some of the results obtained during the realization of the topic of the doctoral dissertation have been published by Tomczyk and Szczerba in [146-151].

1.4. Summary of notations

Throughout the book the index notation is used.

Sub- and superscripts $\alpha, \beta, \ldots$ take the values $1, 2$ and are related to orthonormal parametrization (x^1, x^2) introduced on the shell midsurface; summation convention holds.

Non-tensorial superscripts $A, B, \ldots$ and $a, b, \ldots$ run over sequences $1, 2, \ldots, N$ and $1, 2, \ldots, n$, respectively, where $N \geq 1$ and $n \geq 1$; the summation convention with respect to these indices holds.

The partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Differentiation with respect to time coordinate $t \in [t_0, t_1]$ is represented by the overdot.

Symbol $O(\delta)$, $\delta > 0$, stands for an arbitrary function of δ such that $O(\delta) \neq 0$ for every δ and $\delta \to 0$ implies $O(\delta) \to 0$, $O(\delta)/\delta \to c$ for some $c \neq 0$.

2. Overview of modelling techniques applied to periodic and tolerance-periodic structures

The doctoral thesis deals with micro-heterogeneous cylindrical shells which, on the micro level, consist of a very large number of separated small elements (cells) perfectly bonded to each other and regularly distributed along circumferential direction. The adjacent cells are of nearly identical geometrical and material properties, but the distant elements can be very different. On the macro level, this tolerance-periodic microstructure implies a macroscopically inhomogeneous structures but with a continuous and slow variation of averaged properties. Such shells are called ***functionally graded shells.*** Note, that if the geometrical and material structure of every cell is identical then we deal with periodically heterogeneous shells having constant macroscopic (averaged) properties.

The mechanical/thermal behaviour of periodic or tolerance-periodic (functionally graded) micro-heterogeneous solids (e.g. beams, plates, shells, laminates) is described by means of partial differential equations with periodic or tolerance-periodic, highly oscillating and often discontinuous functional coefficients. Hence, they cannot be directly applied to investigations of engineering problems. To obtain averaged equations with constant or continuously slowly varying coefficients, a lot of different approximate modelling methods for composites of this kind have been proposed. ***We shall restrict our considerations to the best known analytical procedures***. It has to be emphasized that functionally graded structures are often analysed in the framework of averaging approaches similar to those applied to periodic structures, which on the macro level are macroscopically homogeneous.

The overview of modelling techniques used for the analysis of mechanical, thermal or coupled thermo-mechanical problems for periodic as well as functionally graded composite materials and structures can be found in the monographs by Bakhvalov and Panasenko [5], Bensoussan *et al.* [9], Jikov *et al.* [42], Jones [43], Sanchez-Palencia [108], Lewiński and Telega [59], Suresh and Mortensen [110], Nemat-Nasser and Hori [89], Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164] and also in paper by Reiter *et al.* [105].

Mathematical modelling methods related to periodically or locally periodically (called tolerance-periodically in the presented doctoral dissertation) micro-heterogeneous composites can be divided into ***general and special modelling techniques.*** The general modelling methods concern large class of micro-heterogeneous media without the specification of the micro-heterogeneity. On the other hand the special modelling procedures are formulated for certain selected forms of the micro-heterogeneity and can be applied only for some specific forms of coefficients in partial differential equations describing behaviour of the micro-inhomogeneous structures.

We mention here two special modelling techniques: ***the effective stiffness theory***, cf. Herrmann and Achenbach [21], and ***the Floquet-Bloch wave theory***, cf. Stoker [109], Lee [55], which are applied to the analysis of wave propagation problems for one-dimensional periodic structures such as layered media or for periodically and directionally reinforced composites. Both these methods make it possible to take into account the dispersion phenomena in the micro-heterogeneous solids. Some results obtained by applying approaches mentioned above are discussed in Achenbach, Sun and Herrman [2], Herrmann, Kaul and Delph [22], Christensen [13], Kohn, Krumhansl and Lee [50].

Among the *general mathematical modelling methods* we can mention ***homogenization for periodic and local-periodic structures, higher-order homogenization for periodic and local-periodic structures, homogenization by micro-local parameters, tolerance modelling method.***

Periodic/tolerance-periodic structures are usually described using ***homogeneized models derived by means of asymptotic methods***. These models represent certain equivalent structures with constant or continuously slowly varying geometrical and material properties. Homogenization is based on a formal limit passage with length dimensions of a cell to zero. Thus, in the first approximation the homogenized equations neglect the effect of a microstructure size on the macroscopic behaviour of the periodic/tolerance-periodic structures (***the length-scale effect***). This effect plays an important role mainly in the vibration and wave propagation analysis as well as in dynamical stability problems.

The mathematical foundations of this modelling technique can be found in Bakhvalov and Panasenko [5], Bensoussan *et al.* [9], Jikov *et al.* [42], Jones [43], Panasenko [95], Sanchez-Palencia [108]. ***Applications of the asymptotic homogenization procedure to modelling of stationary and non-stationary phenomena for periodically micro-heterogeneous shells (plates) are presented in a large number of contributions***. From the extensive list on this subject we can mention publications by Andrianov *et al.* [4], Caillerie [11], Challagulla *et al.* [12], Georgiades *et al.* [19], Kalamkarov [45], Lewiński [56], Lewiński and Telega [58, 59], Lutoborski [61], Kohn and Vogelius [51]. ***Local-periodic structures, which can be treated as made of functionally graded materials, can also be analysed in the framework of homogenization methods***. Using the known concept of the G-convergence approach, which is a generalization of the homogenized procedure, cf. Jikov *et al.* [42], ***homogenization models for local-periodic structures*** can be derived. These models are used to investigate various problems for functionally graded materials in many papers, e.g. in Miller and Lanutti [83], Itoh, Takahashi and Takano [24]. Some other methods applied in the modelling of functionally graded media are discussed in the paper by Reiter *et al.* [105] and in monographs by Suresh and Mortensen [110], where extensive lists of references can be found. Asymptotic approach to vibrations of functionally graded cylindrical shells can be found in Pradhan *et al.* [101], Rahimi, Ansari and Hemmatnezhad [104], Isvandzibaei, Jamaluddin and Hamzah [23], Young-Wann Kim [169].

As mentioned above, in the first approximation, the homogenized equations neglect the effect of a microstructure size on the macroscopic behaviour of the periodic/tolerance-periodic structures. In order to derive the length-scale models, ***the higher-order homogenization*** has to be applied. The second or higher order approximations must be formulated in the framework of this approach. However, models of this kind have a complicated analytical form. Hence, their applications to the investigations of boundary-value problems often leads to a large number of boundary conditions, which may be not well motivated from the physical viewpoint, cf. Fish and Wen Chen [18], Lewiński and Kucharski [57], where length-scale effect in periodic composites is analysed, Aboudi *et al.* [1], where effects related to microstructure of functionally graded composites are studied.

Homogenization can be also realized using a concept of micro-local parameters. This approach is based on some postulated a priori physical assumptions. Governing equations of the homogenized model are given in terms of the averaged unknown fields and certain extra unknowns called ***microlocal parameters***. Homogenization by micro-local parameters was proposed by Woźniak [160, 161], Matysiak and Woźniak [72]. Averaged models of this kind were applied for solving of various mechanics/thermomechanics problems in a series of papers, e.g. by Matysiak and Nagórko [67, 68] and by Wągrowska [153], where some stationary problems of multilayered elastic plates and of elastic-plastic composites are investigated, respectively; by Matysiak [65], Matysiak and Ukhanska [71], Matysiak and Yevtushenko [73], Matysiak, Pauk and Yevtushenko [69], Matysiak and Perkowski [70] or Kulchytsky-Zhyhailo, Matysiak and Perkowski [54], where heat conduction in periodically layered composites is studied; by Matysiak, Mieszkowski and Perkowski [66], where surface waves in a periodic two-layered half-space are analysed; by Kaczyński and Matysiak [44], where crack problems in micro-periodic reinforced elastic composite are investigated.

The periodically micro-heterogeneous shells (plates) are also modelled as homogeneous orthotropic structures, cf. Ambartsumyan [3], Brush and Almroth [10], Grigoluk and Kabanov [20]. The orthotropic model equations with coefficients independent of the microstructure length parameter can not be used to the analysis of phenomena related to existence of microstructure length-scale effect, e.g. the dispersion of waves, the occurrence of additional higher-order free vibration frequencies and additional higher-order critical forces depending on the cell size.

A new non-asymptotic approach applied to the modelling of mechanical and thermal phenomena in continuum and discrete micro-heterogeneous structures or composite materials was proposed and developed by Woźniak in a large number of papers, e.g. [162, 163] and summarized in the monographs by Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164]. Mathematical formulation of this approach can be also found in Ostrowski [90]. This technique, called ***the tolerance modelling method***, is based on the concept of ***tolerance relations between points and real numbers*** related to the accuracy of the performed manipulations or calculations. The tolerance relations are determined by ***the tolerance parameters.*** The other basic concepts of this procedure are those of ***slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation.*** The tolerance modelling is based on two assumptions. The first of them is called ***the tolerance averaging approximation*** and makes it possible to neglect terms of an order of tolerance parameters. The second one is termed ***the micro-macro decomposition.*** This assumption states that the kinematic or thermal fields occurring in the problem under consideration can be decomposed into *unknown averaged displacements or temperature, slowly-varying in directions of periodicity or tolerant periodicity and highly oscillating fluctuations caused by a periodic or tolerance-periodic structure of the composite medium.* These kinematic/thermal fluctuations are represented by the finite series of products of *the known highly oscillating, periodic/tolerance-periodic fluctuation shape functions and unknown fluctuation amplitudes slowly-varying in periodicity/tolerance-periodicity directions.* The cell-dependent fluctuation shape functions represent either the principal modes of free periodic vibrations of a cell or physically reasonable approximations of these modes. Hence, they can be obtained as solutions to certain periodic eigenvalue problems related to free cell vibrations. In stationary problems, these functions can be treated as the shape functions resulting from the finite element periodic discretization of the cell. The choice of these functions can be also based on the experience or intuition of the researcher. The main modelling concepts and assumptions mentioned above are explained in Chapter 3 of this doctoral thesis. Governing equations of the tolerance models have constant or slowly-varying coefficients depending also on the microstructure size. It means that ***the tolerance modelling makes it possible to analyse the effect of a cell size on the overall behaviour of the composite medium.***

Applications of the tolerance modelling technique to investigations of selected ***elastodynamic and/or stability problems*** for various periodic structures are shown in a large number of contributions, e.g. for lattic-type or cellular media by Cielecka [14], Cielecka, Woźniak Cz. and Woźniak M. [15]; for Euler-Bernoulli-type beams by Mazur-Śniady [74], Mazur-Śniady, Śniady and Zielichowski-Haber [75], Świątek, Jędrysiak and Domagalski [112]; for Kirchoff-type plates by Jędrysiak [25-30], Nagórko and Woźniak [88], Nagórko [84-86]; for Hencky-Bolle-type plates by Baron [6-8], Jędrysiak and Paś [37]; for wavy-type plates by Michalak, Woźniak Cz. and Woźniak M. [82], Michalak [76, 77]; for three-layered plates with inert core by Marczak and Jędrysiak [64]; for periodic cylindrical shells by Tomczyk [115-136], Tomczyk and Woźniak [152], Tomczyk and Mania [145], Tomczyk and Litawska [140-144], Tomczyk *et al.* [137, 138]. Note, that in papers [140-144] the dynamic or stability problems are investigated in the framework of a certain extended version of the classical tolerance modelling technique. This version, proposed by Tomczyk and Woźniak in [152], is based on a new notion of ***weakly slowly-varying function*** being an extension of the known more restrictive concept of slowly-varying functions occurring in the classical tolerance approach. ***General tolerance models*** derived by means of this extended tolerance averaging procedure include a bigger number of length-scale terms than those formulated by applying the classical tolerance modelling.

Elastostatics of thin periodically stiffened plates with moderately large deflections was studied by Domagalski and Gajdzicki [16].

Some thermal/thermal-elasticity problems of micro-periodic composites are also investigated by applying the tolerance modelling technique. Heat conduction problems were studied by Woźniak Cz., Baczyński and Woźniak M. [165], Łaciński and Woźniak [62], Nagórko and Piwowarski [87], Ostrowski and Jędrysiak [91], Ostrowski and Michalak [92], Rychlewska, Szymczyk and Woźniak [106], Wierzbicki and Mazewska [155], Wierzbicki [154], Tomczyk and Gołąbczak [139].

In the last years the tolerance modelling technique is adopted and extended for non-periodic structures, e.g. made of a functionally graded material. Tolerance averaging of equations describing behaviour of those structures leads to equations with continuous and slowly-varying coefficients depending of the microstructure size.

Tolerance models of dynamic or stability problems for different functionally graded structures are proposed in many contributions, e.g. for laminates by Rychlewska and Woźniak [107], Szymczak and Woźniak [111]; for thin Kirchoff-type transversally graded plates by Kaźmierczak and Jędrysiak [47, 48], Jędrysiak and Kaźmierczak-Sobińska [35], Jędrysiak [32-34]; for thin longitudinally graded plates by Michalak [78, 79], Wirowski [156-159], Michalak and Wirowski [81], Jędrysiak and Michalak [36]; for laminated plates by Jędrysiak, Rychlewska and Woźniak [41]; for laminated shells by Woźniak, Rychlewska and Wierzbicki [167]; for thin functionally graded plates with system of ribs by Rabenda and Michalak [102], Michalak and Rabenda [80]; for transversally or longitudinally graded cylindrical shells by Tomczyk and Szczerba [146-151].

Applications of the tolerance averaging technique to analyse heat conduction or thermoelastic problems in functionally graded structures are presented in a large number of papers and books, e.g. for longitudinally graded hollow cylinder by Ostrowski and Michalak [92-94], Ostrowski [90]; for transversally graded laminates by Radzikowska and Jędrysiak [103], Jędrysiak and Radzikowska [39, 40], Pazera and Jędrysiak [99, 100], Jędrysiak and Pazera [38].

Various mechanical and thermal as well as coupled thermo-mechanical problems for functionally graded laminates, thin plates and shallow shells are discussed by Jędrysiak in monograph [31], where the extended list of references on this topic can be found.

It has to be emphasized that by means of a certain formal analytical procedure we can pass directly from the tolerance models to new asymptotic ones. These asymptotic models can be also obtained independently of tolerance modelling by applying ***the consistent asymptotic averaging*** proposed by Woźniak in [164]. In this approach, the kinematic/thermal fields occurring in the problem under consideration are decomposed into highly-oscillating part depending on a certain parameter $\varepsilon \in (0, 1]$ and averaged part independent of this parameter. The highly-oscillating part is determined by the known *fluctuation shape functions*, which similarly as in the tolerance modelling can be obtained as solutions to certain periodic eigenvalue problems or by means of the finite element periodic discretization of the cell or by means of experience/intuition of the researcher. If these functions are not derive as solutions to eigenvalue problems then ***the effective moduli are obtained without specification of the periodic cell problems.*** It is a very important advantage of these asymptotic models because in most cases obtaining the solutions to the cell problems is not easy. This situation is different from that occurring in the known asymptotic homogenization approach, where *only solutions to the periodic cell problems make it possible to define the effective moduli* of the considered structure. It is worth mentioning that not only *the consisten*t but also *the semi-consistent asymptotic modelling procedure* is formulated by Woźniak in monograph [164]. It has to be emphasized that contrary to *consistent asymptotic models equations*, the resulting differential equations of the semi-consistent asymptotic models describe the effect of microstructure size on the overall medium behaviour. Many applications of the asymptotic modelling techniques mentioned above to various problems in microstructured media are shown in [164].

In the presented doctoral thesis, the attention is focused on the continuum mathematical modelling of dynamic problems for tolerance-periodically micro-heterogeneous cylindrical shells. However, some analytical results are compared with numerical those obtained using the commercial computer software Ansys based on the finite element method (FEM). For this reason, it is worth mentioning here papers by Pawlus [96-98], where stability or dynamic problems for annular layered plates with heterogeneous structure in radial or transversal direction are modelled using commercial computer software Abaqus based on FEM. We also mention publications by Kołakowski and Mania [52], Mania [63], Kołakowski and Teter [53], Teter, Mania and Kołakowski [114], Teter, Kołakowski and Mania [113], where static/dynamic buckling/postbuckling problems for functionally graded thin plates/shells subjected to mechanical/thermal/combined dynamic-thermal loads are modelled with application of FEM software Ansys.

3. Concepts and assumptions of the tolerance modelling technique

The partial differential equations or the pertinent integral functionals applied to problems of micro-heterogeneous periodic or tolerance-periodic shells include functional coefficients, which are periodic or tolerance-periodic, highly oscillating and non-continuous. The averaging of these equations or integral functionals realized by using ***the tolerance modelling technique*** leads to *mathematical models with constant or continuously slowly-varying coefficients depending on the microstructure size*, i.e. on the diameter of a basic cell. Hence, the tolerance model equations make it possible to analyse the effect of a cell size on the overall shell behaviour (*the length-scale effect*).

The averaging of the partial differential equations or the pertinent integral functionals with highly oscillating, non-continuous and periodic or tolerance-periodic coefficients can be also realized using asymptotic procedures, cf. Bensoussan *et al.* [9], Jikov *et al.* [42], Woźniak *et al.* (eds.) [164]. Asymptotic modelling used for these equations and integral functionals leads to *mathematical models with constant or continuously slowly varying coefficients*. However, the asymptotic procedures are performed by limit passages with the microstructure length to zero. Hence, the resulting equations are not able to describe the length scale phenomena.

In order to take into account the length-scale effect in dynamic problems for functionally graded shells being object of considerations in the presented doctoral dissertation, the mathematical, non-asymptotic tolerance modelling procedure is applied.

Below, i.e. in Subsections 3.1 and 3.2, *the basic concepts and assumptions of this tolerance modelling technique* are presented, following monographs by Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164], Tomczyk and Woźniak [152], Ostrowski [90].

Moreover, the general line of *the consistent asymptotic modelling procedure* proposed by Woźniak in [164] and applied in this dissertation is outlined in Subsection 3.3.

3.1. Fundamental concepts of the tolerance modelling procedure

The fundamental concepts of the tolerance modelling approach under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation.*

3.1.1 Tolerance relations

The leading role in formulation of the tolerance modelling technique plays the concept of *tolerance relation*. We shall introduce two special tolerance relations.

Tolerance relation between points

Let Ω be a regular region in physical space E^m and λ be a positive real number. Points $\mathbf{x} \equiv (x_1, \ldots, x_m)$, $\mathbf{y} \equiv (y_1, \ldots, y_m)$ belonging to Ω are said to be in tolerance relation determined by λ, $\mathbf{x} \overset{\lambda}{\approx} \mathbf{y}$, if and only if the distance between points $\mathbf{x}$, $\mathbf{y}$ does not exceed λ, i.e. $||\mathbf{x} - \mathbf{y}||_{E^m} \leq \lambda$.

Tolerance relation between real numbers

Let $\widetilde{\delta}$ be a positive real number. Real numbers μ, ν are said to be in tolerance relation determined by $\widetilde{\delta}$, $\mu \overset{\widetilde{\delta}}{\approx} \nu$, if and only if $|\mu - \nu| \leq \widetilde{\delta}$.

Positive parameters λ, $\widetilde{\delta}$ are called ***tolerance parameters***.

Tolerance parameter λ was introduced by Zeeman [170] and the tolerance relation $\overset{\lambda}{\approx}$ was interpreted from a physical point of view as a certain ***indiscernibility relation***. Parameter $\widetilde{\delta}$ introduced by Fichera [17] was physically interpreted as a certain "***upper bound for negligibles***".

3.1.2 Slowly-varying functions

We recall that Ω is a regular region in E^m, points of Ω are denoted by $\mathbf{x} \equiv (x_1, \ldots, x_m)$ and $\mathbf{y} \equiv (y_1, \ldots, y_m)$. Let Ξ be a regular region in E^{n-m} for $n \geq m$. Points of Ξ are denoted by $\xi \equiv (\xi_1, \ldots, \xi_{n-m})$. If $n = m$ then Ξ and ξ drop out from considerations. Let ∂ stand for gradient operator in Ω; $\partial = (\partial_1, \ldots, \partial_m)$, $\partial_i = \partial/\partial x_i$, $i = 1, 2, \ldots, m$. Denote by ∂^k the k-th gradient in Ω.

Define a closed subset Δ of E^m

$$\Delta \equiv [-\lambda_1/2, \lambda_1/2] \times \ldots \times [-\lambda_m/2, \lambda_m/2],$$

where $\lambda_1 > 0, \ldots, \lambda_m > 0$. By λ we denote diameter of Δ, which is assumed to be sufficiently small when compared to the smallest characteristic length dimension of Ω. Let us also denote

$$\Delta(\mathbf{x}) \equiv \mathbf{x} + \Delta, \quad \Omega_\Delta \equiv \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega\}.$$

Subsequently, subset Δ of E^m will be called *the basic cell* with m as a cell dimension. Every $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$, will be referred to *the cell in* Ω *with the centre at* $\mathbf{x}$, Ω_Δ is called the set of all centers.

Let $F(\cdot) \in C^R(\overline{\Omega})$, where $C^R(\overline{\Omega})$ is a space of functions being continuous, bounded and differentiable in $\overline{\Omega} \subset E^m$ together with their gradients up to the R-th order. Nonnegative integer R is assumed to be specified in every problem under consideration. Note, that function F can also depend on $\xi \in \Xi$ (if $n \geq m$) and time coordinate t as parameters. Let us denote by $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$ the set of tolerance parameters. The first of them represents the distances between points in $\overline{\Omega}$, the second one and the k-th one, $k = 1, 2, \ldots, R$, are related to the differences in appropriate space between the values of function $F(\cdot)$ and its gradient $\partial^k F(\cdot)$ in points $\mathbf{x}, \mathbf{y}$ belonging to Ω such that $|\mathbf{x} - \mathbf{y}| \leq \lambda$.

Function $F(\cdot)$ will be referred to as *the slowly-varying of the R-th kind* (with respect to cell Δ and tolerance given by $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$), if and only if the following two conditions are satisfied

$$\begin{aligned} &(i) \quad \big(\forall(\mathbf{x}, \mathbf{y}) \in \Omega^2\big)[(\mathbf{x} \overset{\lambda}{\approx} \mathbf{y}) \Rightarrow F(\mathbf{x}) \overset{\delta_0}{\approx} F(\mathbf{y}) \text{ and} \\ &\qquad\qquad \partial^k F(\mathbf{x}) \overset{\delta_k}{\approx} \partial^k F(\mathbf{y}), \; k = 1, 2, \ldots, R], \\ &(ii) \quad (\forall \mathbf{x} \in \Omega) \left[\lambda \left|\partial^k F(\mathbf{x})\right| \overset{\delta_k}{\approx} 0, \quad k = 1, 2, \ldots, R\right]. \end{aligned} \tag{3.1}$$

Under above conditions we shall write $F \in SV_\delta^R(\Omega, \Delta)$.

In the applications of the tolerance modelling, tolerance parameter λ is known *a priori* as a certain microstructure length, whereas values of tolerance parameters $\delta_0, \delta_1, \ldots, \delta_R$ can be determined only *a posteriori*, i.e. after obtaining solution to the initial-boundary value problem under consideration.

Note, that a new notion of *the weakly slowly-varying function* $F(\cdot)$, $F \in WSV_\delta^R(\Omega, \Delta)$, being an extension of the concept of *slowly-varying function* given above, has been introduced by Tomczyk and Woźniak in [152]. For *the weakly slowly-varying function* the first from conditions (3.1) is satisfied only. Tolerance model equations derived by applying the less restrictive concept of weakly slowly-varying function contain a bigger number of terms depending on a cell size than the model equations obtained by means of slowly-varying function and hence they make it possible to investigate the length-scale effect in more detail.

In the present work the considerations will be based on the notion of *slowly-varying function*.

3.1.3 Tolerance-periodic functions

An essentially bounded and weakly differentiable function f defined on $\overline{\Omega} \in E^m$, which can also depend on $\xi \in \Xi$ (if $n \geq m$) and time coordinate t as parameters, is called *tolerance-periodic of the R-th kind* with respect to cell Δ and tolerance parameters $\delta \equiv (\lambda, \delta_0)$, if for every $\mathbf{x} \in \Omega_\Delta$ there exists Δ-periodic function $\widetilde{f}$ such that $f|\Delta(\mathbf{x}) \cap Dom\ f$ and $\widetilde{f}\,|\Delta(\mathbf{x}) \cap Dom\ \widetilde{f}$ are indiscernible in tolerance determined by $\delta \equiv (\lambda, \delta_0)$. Roughly speaking, function f is *tolerance-periodic* if its values in an arbitrary cell $\Delta(\mathbf{x})$ can be approximated, with sufficient accuracy, by the corresponding values of a certain Δ-periodic function $\widetilde{f}\,(\mathbf{x}, \mathbf{z})$, $\mathbf{z} \in \Delta\,(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$. Function $\widetilde{f}$ is a Δ-*periodic approximation* of f in $\Delta(\mathbf{x})$. For function $f(\cdot)$ being tolerance-periodic together with its derivatives up to the R-th order, we shall write $f \in TP^R_\delta(\Omega, \Delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$. We recall that nonnegative integer R is assumed to be specified in every problem under consideration. It should be noted that for periodic structures function $\widetilde{f}\,(\mathbf{x}, \cdot)$ has the same analytical form in every cell $\Delta(\mathbf{x})$ with a centre at $\mathbf{x} \in \Omega_\Delta$. Hence, $\widetilde{f}$ is independent of $\mathbf{x}$ and we have $\widetilde{f} = \widetilde{f}(\mathbf{z}), \mathbf{z} \in \Delta\,(\mathbf{x})\,, \mathbf{x} \in \Omega_\Delta$. In the general case, i.e. for tolerance-periodic structures (i.e. structures which in small neighbourhoods of $\Delta(\mathbf{x})$ can be approximately regarded as periodic) being objects of considerations in this work, $\widetilde{f}$ depends on $\mathbf{x}$ and hence we have $\widetilde{f} = \widetilde{f}(\mathbf{x}, \mathbf{z}), \mathbf{z} \in \Delta\,(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$.

3.1.4 Fluctuation shape functions

Let h be a tolerance-periodic, highly oscillating function defined in $\overline{\Omega} \in E^m$, which is continuous together with gradients $\partial^k h, k = 1, 2, \ldots, R-1$ and has a piecewise continuous (or in special cases continuous) bounded gradient $\partial^R h$. Tolerance-periodic function $h(\cdot)$ will be called *the fluctuation shape function of the R-th kind*, $h \in FS^R_\delta(\Omega, \Delta)$, if it depends on λ as a parameter and satisfies conditions

$$h \in O\left(\lambda^R\right), \quad \partial^k h \in O\left(\lambda^{R-k}\right), \quad k = 1, 2, \ldots, R, \tag{3.2}$$

$$\int\limits_{\Delta(\mathbf{x})} \widetilde{\mu}(\mathbf{x}, \mathbf{z})\widetilde{h}(\mathbf{x}, \mathbf{z})d\mathbf{z} = 0, \quad \mathbf{x} \in \Omega_\Delta, \tag{3.3}$$

for $\mu(\cdot)$ being a certain positive valued tolerance-periodic function defined in $\overline{\Omega}$. Note, that condition (3.3) holds in dynamic problems. In stationary problems this condition is replaced by

$$\int\limits_{\Delta(\mathbf{x})} \widetilde{h}(\mathbf{x}, \mathbf{z})d\mathbf{z} = 0.$$

Moreover, for every $F \in SV_\delta^R(\Omega, \Delta)$ and $h \in FS_\delta^R(\Omega, \Delta)$, function $\vartheta(\cdot) \equiv h(\cdot)F(\cdot) \in TP_\delta^R(\Omega, \Delta)$ satisfies condition

$$\begin{gathered}\int\limits_{\Delta(\mathbf{x})} \partial^k \widetilde{\vartheta}(\mathbf{x}, \mathbf{z}) d\mathbf{z} = F(\mathbf{x}) \int\limits_{\Delta(\mathbf{x})} \partial^k \widetilde{h}(\mathbf{x}, \mathbf{z}) d\mathbf{z}, \\ k = 0, 1, \ldots, R, \quad \partial^0 \widetilde{\vartheta} \equiv \widetilde{\vartheta}, \quad \partial^0 \widetilde{h} \equiv \widetilde{h}.\end{gathered} \tag{3.4}$$

3.1.5 Averaging operation

Let f be a function defined in $\overline{\Omega} \in E^m$, which is integrable and bounded in every cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$. By *the averaging of* $f(\cdot)$ we shall mean function $\left\langle f(\cdot) \right\rangle (\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$, defined by

$$\langle f \rangle (\mathbf{x}) = \left\langle \widetilde{f} \right\rangle (\mathbf{x}) \equiv \frac{1}{|\Delta|} \int\limits_{\Delta(\mathbf{x})} \widetilde{f}(\mathbf{x}, \mathbf{z}) d\mathbf{z}, \quad \mathbf{x} \in \Omega_\Delta, \tag{3.5}$$

where $\widetilde{f}(\mathbf{x}, \cdot)$ is a periodic approximation of f in $\Delta(\mathbf{x})$. It should be noted that if f is a Δ-periodic function then $\langle f \rangle$ is constant in every Δ. For tolerance-periodic structures, being objects of considerations in this work, $\langle f \rangle (\mathbf{x})$ is a slowly-varying function in $\mathbf{x}$.

3.2. Basic assumptions of the tolerance modelling procedure

The tolerance modelling is based on two assumptions, which are strictly related to the concepts of *the tolerance-periodic*, *slowly-varying* and *fluctuation shape functions*. The first assumption is called *the tolerance averaging approximation.* The second one is termed the *micro-macro decomposition.*

3.2.1 Tolerance averaging approximation

Let f be an arbitrary integrable tolerance-periodic function defined in $\overline{\Omega} \in E^m$, $f \in TP_\delta^R(\Omega, \Delta)$, and let $F \in SV_\delta^R(\Omega, \Delta)$. Moreover, for every $F \in SV_\delta^R(\Omega, \Delta)$ and $h \in FS_\delta^R(\Omega, \Delta)$ we define function $\vartheta(\cdot) \equiv h(\cdot)F(\cdot) \in TP_\delta^R(\Omega, \Delta)$. *The tolerance averaging approximation has the form*

$$\begin{gathered}\left\langle f \partial^k F \right\rangle (\mathbf{x}) = \langle f \rangle (\mathbf{x}) \partial^k F(\mathbf{x}) + O(\delta), \quad k = 0, 1, \ldots, R, \quad \partial^0 F(\mathbf{x}) = F(\mathbf{x}), \\ \langle f \partial^r \vartheta \rangle (\mathbf{x}) = \left\langle f \partial^r (hF) \right\rangle (\mathbf{x}) = \langle f \partial^r h \rangle (\mathbf{x}) F(\mathbf{x}) + O(\delta), \quad r = 1, 2, \ldots, R.\end{gathered} \tag{3.6}$$

In the course of modelling, terms $O(\delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, \ldots, \delta_R)$, will be neglected.

Approximations (3.6) follow directly from conditions (3.1) satisfied by the slowly-varying functions and from condition (3.4) which holds for the fluctuation shape functions.

Let us observe that the slowly-varying functions can be regarded as invariant under averaging.

Approximations given above will be applied in the modelling problems discussed in this dissertation. For details the reader is referred to Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164], Tomczyk and Woźniak [152], Ostrowski [90].

3.2.2 Micro-macro decomposition assumption

The micro-macro decomposition states that the displacements fields, being unknowns of the partial differential equations (or of pertinent integral functionals) describing behaviour of microheterogeneous structure, can be decomposed into *unknown averaged (macroscopic) displacements* being slowly-varying functions in periodicity (or tolerant periodicity) directions and highly oscillating *fluctuations.* Fluctuations of displacements are represented by the *known highly oscillating Δ-periodic or tolerance-periodic fluctuation shape functions multiplied by unknown fluctuation (microscopic) amplitudes* slowly-varying in periodicity (or tolerant periodicity) directions.

Micro-macro decompositions introduced in the problems discussed in this doctoral dissertation are presented in Subsections 5.1 and 5.3.

3.3. Basic concepts and assumptions of the consistent asymptotic modelling procedure

The fundamental concepts of the consistent asymptotic procedure are those of *the fluctuation shape functions and the averaging operation.* These concepts have been explained in Subsection 3.1. It means that the consistent asymptotic modelling does not require notions of tolerance-periodic and slowly-varying functions.

The fundamental assumption imposed on the starting lagrangian under consideration in the framework of the asymptotic approach is called *the consistent asymptotic decomposition.* It states that the displacement fields occurring in the lagrangian have to be replaced by families of fields depending on parameter $\varepsilon \in (0, 1]$ and defined in an arbitrary cell. These families of displacements are decomposed into averaged part independent of ε and highly-oscillating part depending on ε.

Consistent asymptotic decomposition introduced in problems discussed in this doctoral dissertation is presented in Subsection 5.2.

4. Formulation of the modelling problem

4.1. Thin cylindrical shells with a tolerance-periodic microstructure and a functionally graded macrostructure

Thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells with a tolerance-periodic microstructure in circumferential direction are analysed. It means that ***on the microscopic level***, the shells under consideration consist of many separated, small elements regularly distributed along circumferential direction and perfectly bonded to each other (or to the homogeneous matrix). These elements, called *cells*, are treated as thin shells. It is assumed that ***the adjacent cells are nearly identical*** (*i.e. they have nearly the same geometrical, elastic and inertial properties*), ***but the distant elements can be very different***. As examples we can mention cylindrical shells made of two kinds of tolerance-periodically distributed materials as shown in Fig. 4.1a or shells with tolerance-periodically spaced stiffeners as shown in Fig. 4.2a. Note, that the ribbed shell shown in Fig. 4.2 can be treated on the micro-level as a shell with not only tolerance-periodically distributed elastic and inertial properties, but also with tolerance-periodically distributed geometrical properties. At the same time, the shells considered here have constant structure in axial direction. On the microscopic level, the geometrical, elastic and inertial properties of these shells are determined by highly oscillating, non-continuous and *tolerance-periodic functions* in circumferential direction.

On the other hand, ***on the macroscopic level***, the averaged (macroscopic) properties of the shells are described by functions being *continuous* and *slowly varying* along circumferential direction. It means that on the macro-level the tolerance-periodic shells under consideration can be treated as made of *functionally graded materials* (FGM), cf. Suresh and Mortensen [110], and called *functionally graded shells*, cf. Figs. 4.1b, 4.2b. Moreover, since effective properties of the shells are graded in direction normal to interfaces between constituents, this gradation is referred to as ***the transversal gradation***.

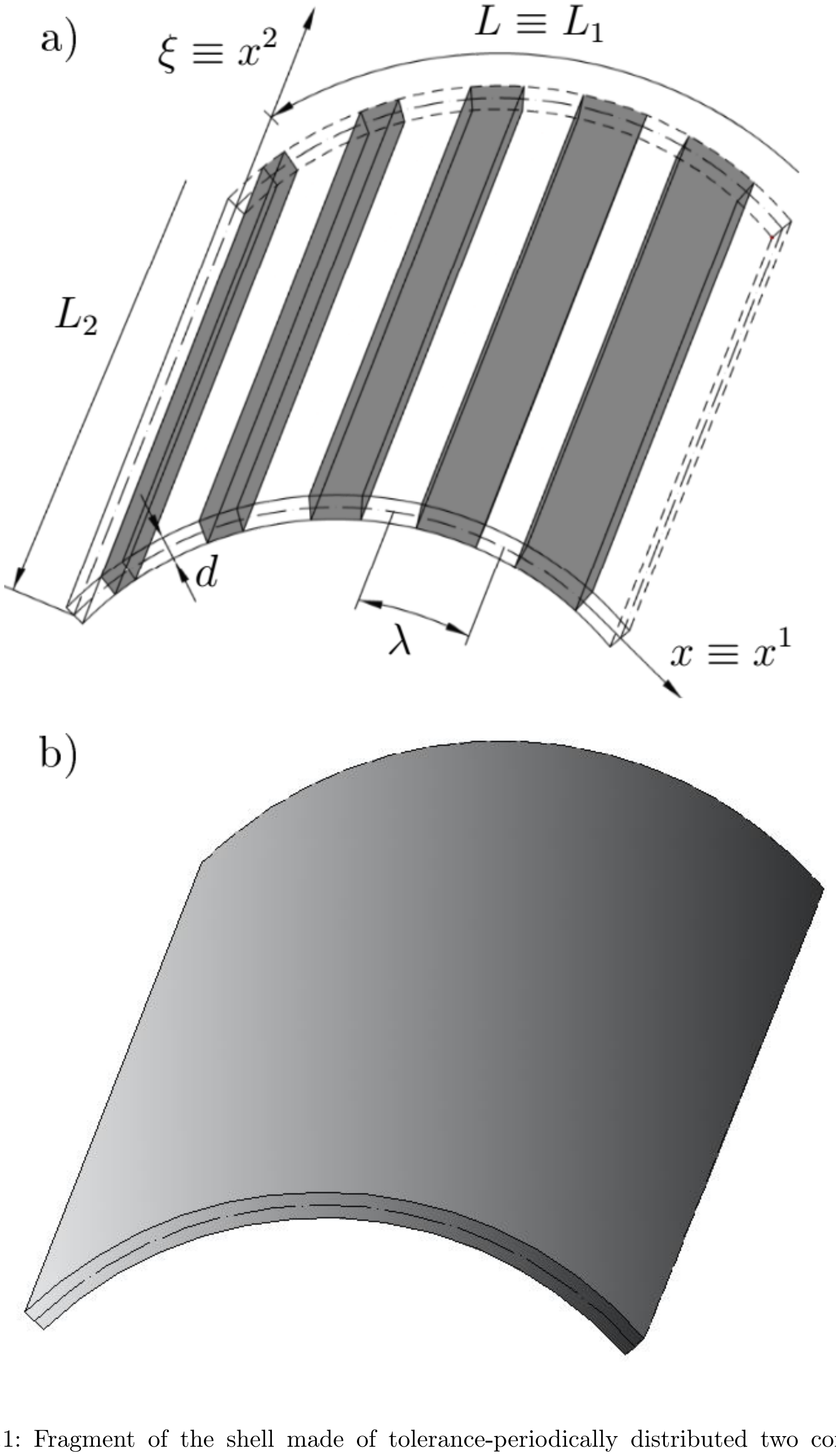

Figure 4.1: Fragment of the shell made of tolerance-periodically distributed two component materials: a) the microscopic point of view b) the macroscopic point of view

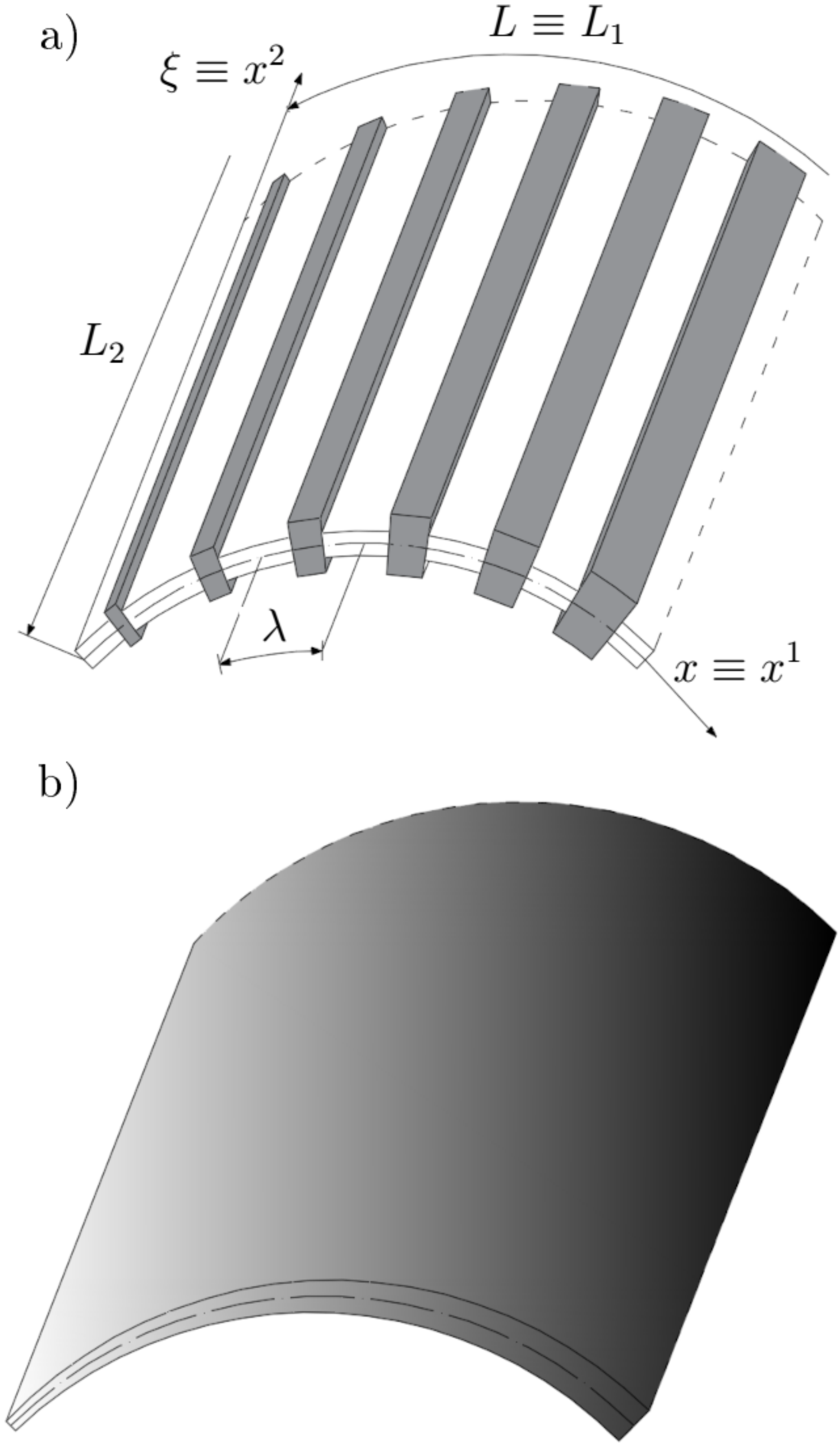

Figure 4.2: Fragment of the shell with two families of tolerance-periodically spaced stiffeners: a) the microscopic point of view b) the macroscopic point of view

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Figs. 4.1 and 4.2. Let $O\overline{x}^1\overline{x}^2\overline{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space E^3 and denote $\overline{\mathbf{x}} \equiv (\overline{x}^1, \overline{x}^2, \overline{x}^3)$. A cylindrical shell midsurface M is given by $M \equiv \left\{\overline{\mathbf{x}} \in E^3 : \overline{\mathbf{x}} = \overline{\mathbf{r}}\left(x^1, x^2\right), \left(x^1, x^2\right) \in \Omega \times \Xi\right\}$, where $\overline{\mathbf{r}}(\cdot)$ is the smooth invertible function such that $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 0$, $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^1 = 1$, $\partial\overline{\mathbf{r}}/\partial x^2 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 1$. It means that on M we have introduced *the orthonormal parametrization*, cf. Fig. 4.3. Note, that derivative $\partial\overline{\mathbf{r}}/\partial x^\alpha$, $\alpha = 1, 2$, should be understood as differentiation of each component of $\overline{\mathbf{r}} \in E^3$, i.e. $\partial\overline{\mathbf{r}}/\partial x^\alpha = \left[\partial\overline{r}^1/\partial x^\alpha, \partial\overline{r}^2/\partial x^\alpha, \partial\overline{r}^3/\partial x^\alpha\right]$.

Sub- and superscripts $\alpha, \beta, \ldots$ run over sequence $1, 2$ and are related to midsurface parameters x^1, x^2; summation convention holds. Partial differentiation related to x^α is represented by ∂_α, i.e. $\partial_\alpha = \partial/\partial x^\alpha$. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Differentiation with respect to time coordinate $t \in \mathrm{I} \equiv [t_0, t_1]$ is represented by the overdot.

Denote by $a_{\alpha\beta}$ and $a^{\alpha\beta}$ the covariant and contravariant midsurface first metric tensors, respectively. Under orthonormal parametrization introduced on M, $a_{\alpha\beta} = a^{\alpha\beta}$ are the unit tensors. Let $b_{\alpha\beta}$ stand for the covariant midsurface second metric tensor. For the introduced parametrization $b_{22} = b_{12} = b_{21} = 0$ and $b_{11} = -r^{-1}$.

Let $d(x)$ and r stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ and an arbitrary cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ (Ω_Δ is a set of all cell centres) are defined by means of

$$
\begin{aligned}
&\Delta \equiv \left[-\lambda/2, \lambda/2\right], \\
&\Delta(x) \equiv x + \Delta = \left[x - \lambda/2, x + \lambda/2\right], \\
&x \in \Omega_\Delta, \ \Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\},
\end{aligned}
\tag{4.1}
$$

where λ is a cell length dimension in $x \equiv x^1$-direction, cf. Figs. 4.1 and 4.2. *The microstructure length parameter* λ satisfies conditions: $\lambda/\max(d) \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$.

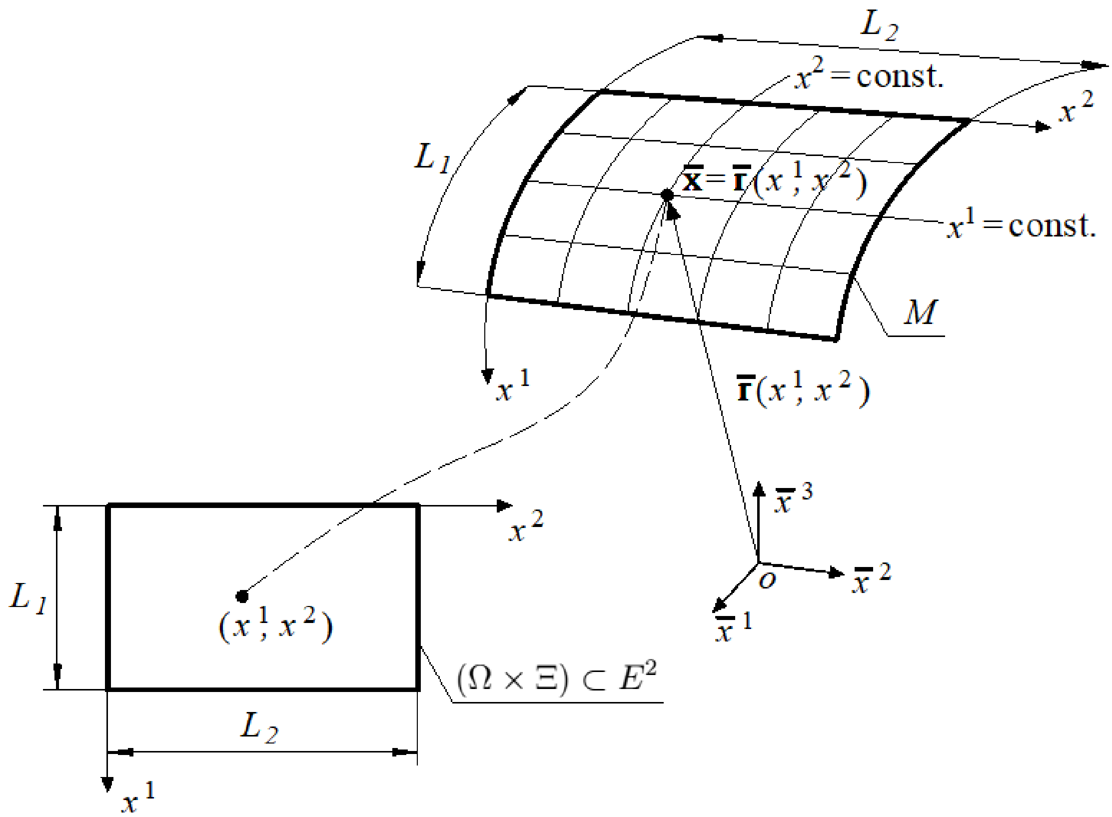

Figure 4.3: Parametrization of the shell midsurface

Setting $z \equiv z^1 \in \left[-\lambda/2, \lambda/2\right]$, we assume that the cell Δ has a symmetry axis for $z = 0$. It is also assumed that inside the cell the geometrical, elastic and inertial properties of the shell are described by symmetric (i.e. even) functions of argument z. At the same time, these functions are independent of argument $\xi \equiv x^2 \in \Xi$.

4.2. Fundamental equations

Denote by $u_\alpha = u_\alpha(x, \xi, t)$, $w = w(x, \xi, t)$, $x \equiv x^1 \in \Omega \equiv (0, L_1)$, $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, $t \in \mathrm{I} \equiv [t_0, t_1]$, the shell displacements in the directions tangent and normal to M, respectively. Elastic properties of the shell are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. Let $f^\alpha(x, \xi, t)$, $f(x, \xi, t)$ be external forces per midsurface unit area, respectively tangent and normal to M.

The considerations will be based on the well-known simplified linear Kirchhoff-Love theory of thin elastic shells in which terms depending on the midsurface second metric tensor $b_{\alpha\beta}$ are neglected in the formulae for curvature changes, cf. Kaliski [46]. In the framework of the shell theory under consideration, strain energy function $E(x,\xi,t)$, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$, related to midsurface M has the form

$$E=\frac{1}{2}\left(D^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}+B^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta}\right), \tag{4.2}$$

where the membrane $\varepsilon=\varepsilon_{\alpha\beta}(x,\xi,t)$ and curvature $\kappa=\kappa_{\alpha\beta}(x,\xi,t)$, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$, strain tensors are

$$\varepsilon_{\alpha\beta}=\frac{1}{2}(\partial_\beta u_\alpha+\partial_\alpha u_\beta)-b_{\alpha\beta}w,\quad \kappa_{\alpha\beta}=-\partial_{\alpha\beta}w. \tag{4.3}$$

The kinetic energy function $K=K(x,\xi,t)$ related to midsurface M and the potential of external loadings $F=F(x,\xi,t)$, $(x,\xi,t)\in\Omega\times\Xi\times\mathrm{I}$, for the shell under consideration are given by

$$K=\frac{1}{2}\mu(\dot u_\alpha\dot u_\beta a^{\alpha\beta}+\dot w\dot w), \tag{4.4}$$

$$F=f^\alpha u_\alpha+fw. \tag{4.5}$$

We recall that $a^{\alpha\beta}$ in (4.4) is the first metric tensor of the shell midsurface M, which under orthonormal parametrization introduced on M is a unit tensor.

Let us introduce the action functional

$$A(u_\alpha,w)=\int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1}L(x,\xi,t,\partial_\beta u_\alpha,u_\alpha,\dot u_\alpha,\partial_{\alpha\beta}w,w,\dot w)dtd\xi dx, \tag{4.6}$$

with lagrangian L being a highly oscillating function with respect to x, $x\in\Omega$. Lagrangian L has the well-known form

$$L=K-E+F, \tag{4.7}$$

where kinetic energy K, strain energy E and potential of external loadings F are given above.

Substituting (4.2)-(4.5) into (4.7) and taking into account that for the parametrization introduced on the shell midsurface M, the components $b_{\alpha\beta}$ of the second metric tensor of M are $b_{22}=b_{12}=b_{21}=0$ and $b_{11}=-r^{-1}$, we arrive at Lagrange function (4.7) in the form

$$\begin{aligned}L=-\frac{1}{2}\Big(&D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha\partial_\delta u_\gamma+\frac{2}{r}D^{\alpha\beta 11}w\partial_\beta u_\alpha+\frac{1}{r^2}D^{1111}ww+\\&+B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w-\mu a^{\alpha\beta}\dot u_\alpha\dot u_\beta-\mu\dot w^2\Big)+f^\alpha u_\alpha+fw.\end{aligned} \tag{4.8}$$

The principle of stationary action applied to A (4.6) leads to the following system of Euler-Lagrange equations

$$\begin{aligned}
&\partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} - \frac{\partial L}{\partial u_\alpha} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_\alpha} = 0,\\
&-\partial_{\alpha\beta} \frac{\partial L}{\partial(\partial_{\alpha\beta} w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 0,
\end{aligned} \tag{4.9}$$

From equations (4.9) combined with (4.8), we obtain *the fundamental equations of the shell theory under consideration* in the explicit form

$$\begin{aligned}
&\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta 11}w) - \mu a^{\alpha\beta}\ddot{u}_\beta + f^\alpha = 0,\\
&r^{-1}D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) + r^{-2}D^{1111}w + \mu\ddot{w} - f = 0.
\end{aligned} \tag{4.10}$$

In the above equations the displacements $u_\alpha(x,\xi,t)$, $w(x,\xi,t)$, $(x,\xi,t) \in \Omega\times\Xi\times\mathrm{I}$, are the basic unknowns. For tolerance-periodic shells, coefficients $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $x \in \Omega$, of lagrangian L and hence also of equations (4.10) are tolerance-periodic, highly oscillating and non-continuous functions with respect to x. That is why obtaining the exact analytical solutions to initial/boundary value problem for Euler-Lagrange equations (4.9) or for their explicit form (4.10) in the most cases is not possible and also numerical problems for these equations are ill conditioned. The first aim of this dissertation is to "replace" these equations by equations with continuous and slowly-varying coefficients depending also on microstructure size λ. To this end ***the non-asymptotic tolerance modelling technique*** and ***the consistent asymptotic modelling procedure*** will be applied to action functional (4.6) determined by Lagrange function (4.8). We recall that to make the analysis more clear, in Chapter 3 we outlined the basic concepts and the main assumptions of these modelling techniques following books by Woźniak and Wierzbicki [168], Woźniak, Michalak and Jędrysiak (eds.) [166], Woźniak *et al.* (eds.) [164], Tomczyk and Woźniak [152] and Ostrowski [90].

5. Averaged models

In this chapter three averaged models of dynamic problems for the thin transversally graded cylindrical shells under consideration will be derived:

- the tolerance model,
- the consistent asymptotic model,
- the combined asymptotic-tolerance model.

5.1. Tolerance model

In this subsection *a new mathematical non-asymptotic averaged model for the analysis of selected dynamic problems for thin shells with a tolerance-periodic microstructure and a functionally (transversally) graded macrostructure in the circumferential direction*, cf. Figs. 4.1 and 4.2, will be derived applying *the tolerance modelling technique.*

We recall that for the shells under consideration, we defined a bounded domain $\Omega \times \Xi$ by means of $\Omega \equiv (0, L_1) \subset E^1$, $\Xi \equiv (0, L_2) \subset E^1$. Points of Ω and Ξ are denoted respectively by $x \equiv x^1$ and $\xi \equiv x^2$. We also defined the basic cell as $\Delta \equiv \left[-\lambda/2, \lambda/2\right]$, where $\lambda \equiv \lambda_1$ is a cell length dimension in x-direction and is called the microstructure length parameter, cf. Chapter 4.

We recall that for the considered shells, coefficients in the fundamental equations (4.10) are tolerance-periodic, highly oscillating and non-continuous functions in $x \in \Omega$. At the same time, these coefficients are independent of argument $\xi \in \Xi$.

The tolerance modelling procedure for Euler-Lagrange equations (4.9) is realized in two steps.

The first step is *the tolerance averaging of lagrangian* (4.8). To this end let us introduce two systems of linear-independent highly oscillating *fluctuation shape functions*, being tolerance-periodic in x: $h^a \in FS^1_\delta(\Omega, \Delta)$, $a = 1, 2, \ldots, n$ and $g^A \in FS^2_\delta(\Omega, \Delta)$, $A = 1, 2, \ldots, N$. These functions are assumed to be known in every problem under consideration. They represent oscillations inside a cell. The functions depend on λ as parameter and agree with (3.2) and (3.3) they have to satisfy conditions

$$h^a \in O(\lambda),\ \lambda\partial_1 h^a \in O(\lambda),$$
$$g^A \in O(\lambda^2),\ \lambda\partial_1 g^A \in O(\lambda^2),\ \lambda^2\partial_{11} g^A \in O(\lambda^2),$$
$$\langle \mu h^a \rangle = \left\langle \mu g^A \right\rangle = 0 \text{ and } \left\langle \mu h^a h^b \right\rangle = \left\langle \mu g^A g^B \right\rangle = 0 \text{ for } a \neq b, A \neq B,$$

where $\mu(x)$ is the shell mass density being a tolerance-periodic function with respect to $x \in \Omega$.

Taking into account that inside the cell the geometrical, elastic and inertial properties of the shells under consideration are described by symmetric (i.e. even) functions of argument $z \equiv z^1 \in \Delta(x)$ (the cell has a symmetry axis for $z = 0$), we assume that periodic approximation $\widetilde{h}^a(x,z)$ of $h^a(x)$ in $\Delta(x)$, $x \in \Omega_\Delta$ is either even or odd function with respect to z. This same restriction is imposed on periodic approximation $\widetilde{g}^A(x,z)$ of fluctuation shape function $g^A(x)$. Let $\varphi \in TP_\delta^R(\Omega,\Delta)$, be an even function with respect to $z \in \Delta(x)$. Under aforementioned restriction, averages $\langle \varphi h \partial_1 h \rangle$, $\langle \varphi g \partial_1 g \rangle$, $\langle \varphi \partial_1 g \partial_{11} g \rangle$, which appear in the course of modelling of tolerance-periodic shells are equal to zero.

Now, we have to introduce *the micro-macro decomposition* of displacements $u_\alpha(x,\xi,t)$, $u_\alpha \in TP_\delta^1(\Omega,\Delta)$, $w(x,\xi,t)$, $w \in TP_\delta^2(\Omega,\Delta)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, which in the problem under consideration is assumed in the form

$$\begin{aligned} u_\alpha(x,\xi,t) &= u_\alpha^0(x,\xi,t) + h^a(x) U_\alpha^a(x,\xi,t), \quad a = 1,2,\ldots,n, \\ w(x,\xi,t) &= w^0(x,\xi,t) + g^A(x) W^A(x,\xi,t), \quad A = 1,2,\ldots,N, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} &u_\alpha^0(\cdot,\xi,t), U_\alpha^a(\cdot,\xi,t) \in SV_\delta^1(\Omega,\Delta), \quad \delta \equiv (\lambda,\delta_0,\delta_1), \\ &w^0(\cdot,\xi,t), W^A(\cdot,\xi,t) \in SV_\delta^2(\Omega,\Delta), \quad \delta \equiv (\lambda,\delta_0,\delta_1,\delta_2), \end{aligned} \tag{5.2}$$

for every $(\xi,t) \in \Xi \times \mathrm{I}$, summation convention over a and A holds. Functions u_α^0, w^0 called ***averaged*** *(macroscopic)* ***variables*** (or ***macrodisplacements***) and functions U_α^a, W^A termed ***fluctuation*** *(microscopic)* ***amplitudes*** are ***the new unknowns slowly-varying in*** $x \in \Omega$.

Finite sums $h^a(x) U_\alpha^a(x,\xi,t)$, $a = 1,2,\ldots,n$ and $g^A(x) W^A(x,\xi,t)$, $A = 1,2,\ldots,N$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, represent oscillations of displacements caused by a tolerance-periodic microheterogeneous structure of the shell. Integers n, N determine accuracy of solutions to the initial-boundary value problems under consideration.

We substitute the right-hand sides of (5.1) into (4.8). The resulting lagrangian is denoted by L_{hg}. Averaging lagrangian L_{hg} over cell Δ using averaging formula (3.5) and *tolerance averaging approximation* (3.6), we arrive at function $\langle L_{hg} \rangle$ being *the tolerance averaging of lagrangian* (4.8) *in* Δ *under micro-macro decomposition* (5.1). The obtained result has the form

$$
\begin{aligned}
&\langle L_{hg}\rangle \Big(x, \partial_\beta u^0_\alpha, u^0_\alpha, \partial_2 U^a_\alpha, U^a_\alpha, \dot{u}^0_\alpha, \dot{U}^a_\alpha, \partial_{\alpha\beta} w^0,\\
&w^0, \partial_{22} W^A, \partial_2 W^A, W^A, \dot{w}^0, \dot{W}^A\Big) =\\
&= -\frac{1}{2}\Big[\Big\langle D^{\alpha\beta\gamma\delta}\Big\rangle \partial_\beta u^0_\alpha \partial_\delta u^0_\gamma + 2\Big\langle D^{\alpha\beta\gamma 1}\partial_1 h^a\Big\rangle \partial_\beta u^0_\alpha U^a_\gamma +\\
&+ 2\underline{\Big\langle D^{\alpha\beta\gamma 2} h^a\Big\rangle}\partial_\beta u^0_\alpha \partial_2 U^a_\gamma + \Big\langle D^{\alpha 11\gamma}\partial_1 h^a \partial_1 h^b\Big\rangle U^a_\gamma U^b_\alpha +\\
&+ \underline{\Big\langle D^{\alpha 22\gamma} h^a h^b\Big\rangle}\partial_2 U^b_\gamma \partial_2 U^a_\alpha + 2r^{-1}\Big(\Big\langle D^{\alpha\beta 11}\Big\rangle \partial_\beta u^0_\alpha w^0 +\\
&+ \langle D^{\alpha 111}\partial_1 h^a\rangle w^0 U^a_\alpha + \underline{\Big\langle D^{\alpha\beta 11} g^A\Big\rangle}\partial_\beta u^0_\alpha W^A + \underline{\Big\langle D^{\alpha 111}\partial_1 h^a g^A\Big\rangle} U^a_\alpha W^A +\\
&+ \underline{\langle D^{\alpha 211} h^a\rangle}\partial_2 U^a_\alpha w^0 + \underline{\Big\langle D^{\alpha 211} h^a g^A\Big\rangle}\partial_2 U^a_\alpha W^A\Big) + r^{-2}\Big(\langle D^{1111}\rangle w^0 w^0 +\\
&+ 2\underline{\Big\langle D^{1111} g^A\Big\rangle} w^0 W^A + \underline{\Big\langle D^{1111} g^A g^B\Big\rangle} W^A W^B\Big) + \Big\langle B^{\alpha\beta\gamma\delta}\Big\rangle \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 +\\
&+ 2\Big(\Big\langle B^{\alpha\beta 11}\partial_{11} g^A\Big\rangle \partial_{\alpha\beta} w^0 W^A + \underline{\Big\langle B^{\alpha\beta 22} g^A\Big\rangle}\partial_{\alpha\beta} w^0 \partial_{22} W^A +\\
&+ \underline{\Big\langle B^{1122} g^A \partial_{11} g^B\Big\rangle}\partial_{22} W^B W^A\Big) + 4\underline{\Big\langle B^{\alpha\beta 12}\partial_1 g^A\Big\rangle}\partial_{\alpha\beta} w^0 \partial_2 W^A +\\
&+ 4\underline{\Big\langle B^{1212}\partial_1 g^A \partial_1 g^B\Big\rangle}\partial_2 W^A \partial_2 W^B + \Big\langle B^{1111}\partial_{11} g^A \partial_{11} g^B\Big\rangle W^A W^B +\\
&+ \underline{\Big\langle B^{2222} g^A g^B\Big\rangle}\partial_{22} W^A \partial_{22} W^B - \langle\mu\rangle a^{\alpha\beta}\dot{u}^0_\alpha \dot{u}^0_\beta - \langle\mu\rangle \left(\dot{w}^0\right)^2 +\\
&- \underline{\Big\langle \mu h^a h^b\Big\rangle} a^{\alpha\beta}\dot{U}^a_\alpha \dot{U}^b_\beta - \underline{\Big\langle \mu g^A g^B\Big\rangle}\dot{W}^A \dot{W}^B\Big] + \langle f^\alpha\rangle u^0_\alpha +\\
&+ \underline{\langle f^\alpha h^a\rangle} U^a_\alpha + \langle f\rangle w^0 + \underline{\Big\langle f g^A\Big\rangle} W^A.
\end{aligned}
\tag{5.3}
$$

The underlined terms in (5.3) depend on microstructure length parameter λ. Action functional

$$
A_{hg}(u^0_\alpha, U^a_\alpha, w^0, W^A) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} \langle L_{hg}\rangle \, dt d\xi dx, \tag{5.4}
$$

with $\langle L_{hg}\rangle$ given by (5.3), is called ***the tolerance averaging of action functional*** $A(u_\alpha, w)$ defined by (4.6) *under decomposition* (5.1).

The second step in the tolerance modelling of Euler-Lagrange equations (4.9) is to apply the principle of stationary action to A_{hg} given above.

Under assumption that $\partial\langle L_{hg}\rangle/\partial_\beta u^0_\alpha$, $\langle\partial L_{hg}\rangle/\partial_{\alpha\beta} w^0$, $\partial\langle L_{hg}\rangle/\partial_2 U^a_\alpha$, $\partial\langle L_{hg}\rangle/\partial_2 W^A$, $\partial\langle L_{hg}\rangle/\partial_{22} W^A$ are continuous, from the principle of stationary action applied to A_{hg} we obtain the following system of Euler-Lagrange equations for u^0_α, w^0, U^a_α, W^A as the basic unknowns

$$\begin{aligned}
&\partial_\beta \frac{\partial\langle L_{hg}\rangle}{\partial\left(\partial_\beta u^0_\alpha\right)} - \frac{\partial\langle L_{hg}\rangle}{\partial u^0_\alpha} + \frac{\partial}{\partial t}\frac{\partial\langle L_{hg}\rangle}{\partial \dot{u}^0_\alpha} = 0,\\
&-\partial_{\alpha\beta} \frac{\partial\langle L_{hg}\rangle}{\partial\left(\partial_{\alpha\beta} w^0\right)} - \frac{\partial\langle L_{hg}\rangle}{\partial w^0} + \frac{\partial}{\partial t}\frac{\partial\langle L_{hg}\rangle}{\partial \dot{w}^0} = 0,\\
&\frac{\partial}{\partial t}\frac{\partial\langle L_{hg}\rangle}{\partial \dot{U}^a_\alpha} - \frac{\partial\langle L_{hg}\rangle}{\partial U^a_\alpha} + \partial_2 \frac{\partial\langle L_{hg}\rangle}{\partial\left(\partial_2 U^a_\alpha\right)} = 0,\\
&\frac{\partial}{\partial t}\frac{\partial\langle L_{hg}\rangle}{\partial \dot{W}^A} - \frac{\partial\langle L_{hg}\rangle}{\partial W^A} + \partial_2 \frac{\partial\langle L_{hg}\rangle}{\partial\left(\partial_2 W^A\right)} - \partial_{22} \frac{\partial\langle L_{hg}\rangle}{\partial\left(\partial_{22} W^A\right)} = 0.
\end{aligned} \tag{5.5}$$

Combining (5.5) with (5.3) we arrive finally at the explicit form of ***the tolerance model equations under micro-macro decomposition*** (5.1). We shall write these equations in the form of

- the constitutive equations

$$\begin{aligned}
N^{\alpha\beta} &= \left\langle D^{\alpha\beta\gamma\delta}\right\rangle \partial_\delta u^0_\gamma + r^{-1}\left(\left\langle D^{\alpha\beta 11}\right\rangle w^0 + \underline{\left\langle D^{\alpha\beta 11} g^B\right\rangle} W^B\right) +\\
&+ \left\langle D^{\alpha\beta\gamma 1}\partial_1 h^b\right\rangle U^b_\gamma + \underline{\left\langle D^{\alpha\beta\gamma 2} h^b\right\rangle}\partial_2 U^b_\gamma,\\
M^{\alpha\beta} &= \left\langle B^{\alpha\beta\gamma\delta}\right\rangle \partial_{\gamma\delta} w^0 + \left\langle B^{\alpha\beta 11}\partial_{11} g^B\right\rangle W^B +\\
&+ 2\underline{\left\langle B^{\alpha\beta 12}\partial_1 g^B\right\rangle}\partial_2 W^B + \underline{\left\langle B^{\alpha\beta 22} g^B\right\rangle}\partial_{22} W^B,\\
H^{a\beta} &= \left\langle \partial_1 h^a D^{\beta 1\gamma\delta}\right\rangle \partial_\delta u^0_\gamma - \underline{\left\langle h^a D^{\beta 2\gamma\delta}\right\rangle}\partial_{2\delta} u^0_\gamma +\\
&+ \left\langle \partial_1 h^a D^{\beta 11\gamma}\partial_1 h^b\right\rangle U^b_\gamma - \left\langle h^a D^{\beta 22\gamma} h^b\right\rangle \partial_{22} U^b_\gamma +\\
&+ r^{-1}\left(\left\langle \partial_1 h^a D^{\beta 111}\right\rangle w^0 - \underline{\left\langle h^a D^{\beta 211}\right\rangle}\partial_2 w^0 +\right.\\
&+ \left.\underline{\left\langle \partial_1 h^a D^{\beta 111} g^B\right\rangle} W^B - \underline{\left\langle h^a D^{\beta 211} g^B\right\rangle}\partial_2 W^B\right),
\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
G^A = r^{-1} \Big(& \underline{\left\langle g^A D^{11\gamma\delta} \right\rangle} \partial_\delta u^0_\gamma + \underline{\left\langle \partial_1 h^b D^{111\gamma} g^A \right\rangle} U^b_\gamma + \\
& + \underline{\left\langle h^b D^{112\gamma} g^A \right\rangle} \partial_2 U^b_\gamma \Big) + r^{-2} \underline{\left\langle g^A D^{1111} \right\rangle} w^0 + \\
& + \left\langle \partial_{11} g^A B^{11\alpha\beta} \right\rangle \partial_{\alpha\beta} w^0 - 2 \underline{\left\langle \partial_1 g^A B^{\alpha\beta 12} \right\rangle} \partial_{\alpha\beta 2} w^0 + \\
& + \underline{\left\langle g^A B^{\alpha\beta 22} \right\rangle} \partial_{\alpha\beta 22} w^0 + \Big(\left\langle \partial_{11} g^A B^{1111} \partial_{11} g^B \right\rangle + \\
& + r^{-2} \underline{\left\langle g^A D^{1111} g^B \right\rangle} \Big) W^B + \Big(\underline{\left\langle \partial_{11} g^A B^{1122} g^B \right\rangle} + \\
& + \underline{\left\langle g^A B^{1122} \partial_{11} g^B \right\rangle} - 4 \underline{\left\langle \partial_1 g^A B^{1212} \partial_1 g^B \right\rangle} \Big) \partial_{22} W^B + \\
& + \underline{\left\langle g^A B^{2222} g^B \right\rangle} \partial_{2222} W^B,
\end{aligned}
\tag{5.6$_{contd}$}
$$

- and the dynamic equilibrium equations for unknowns u^0_α, w^0, U^a_α, W^A being slowly-varying functions with respect to $x \in \Omega$

$$
\begin{aligned}
& \partial_\alpha N^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \ddot{u}^0_\alpha + \langle f^\beta \rangle = 0, \\
& \partial_{\alpha\beta} M^{\alpha\beta} + r^{-1} N^{11} + \langle \mu \rangle \ddot{w}^0 - \langle f \rangle = 0, \\
& \underline{\langle \mu h^a h^b \rangle} a^{\alpha\beta} \ddot{U}^b_\alpha + H^{a\beta} - \underline{\langle f^\beta h^a \rangle} = 0, \quad a, b = 1, 2, \ldots, n, \\
& \underline{\langle \mu g^A g^B \rangle} \ddot{W}^B + G^A - \underline{\langle f g^A \rangle} = 0, \quad A, B = 1, 2, \ldots, N.
\end{aligned}
\tag{5.7}
$$

Equations (5.6) and (5.7) together with *micro-macro decomposition* (5.1) and *physical reliability conditions* (5.2) constitute *the tolerance model of selected dynamic problems for the thin transversally graded shells under consideration.*

Discussion of results

The characteristic features of the derived *tolerance model* are:

- In contrast to starting equations (4.10) with discontinuous, highly oscillating and tolerance-periodic coefficients, the tolerance model equations (5.6) and (5.7) proposed here *have coefficients being continuous and slowly-varying functions in x, $x \in \Omega$*. Moreover, *some of them depend on microstructure length parameter λ* (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of length scale on the global shell behaviour.* Moreover, we can analyse the length-scale effect not only in dynamic but also in stationary problems for the transversally graded shells.

- Macrodisplacements u^0_α, w^0 are governed by the system of three partial differential equations $(5.7)_{1,2}$. The number and form of boundary conditions for averaged variables u^0_α, w^0 are the same as in the classical shell theory governed by equations (4.10). Fluctuation amplitudes U^a_α, W^A, $a = 1, 2, \ldots, n$, $A = 1, 2, \ldots, N$ are governed by the system of $(2n+N)$ partial differential equations $(5.7)_{3,4}$. The boundary conditions for U^a_α, W^A should be defined only on boundaries $\xi = 0$, $\xi = L_2$.

- Decomposition (5.1) and hence also resulting tolerance model equations (5.6) and (5.7) are uniquely determined by the postulated *a priori* tolerance-periodic *fluctuations shape functions*, $h^a \in FS^1_\delta(\Omega, \Delta)$, $h^a \in O(\lambda)$, $a = 1, 2, \ldots, n$ and $g^A \in FS^2_\delta(\Omega, \Delta)$, $g^A \in O(\lambda^2)$, $A = 1, 2, \ldots, N$ which represent oscillations inside a cell. These functions can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf. Tomczyk [133], Jędrysiak [27]. It means that they represent either the principal modes of free periodic vibrations of the cell or physically reasonable approximations of these modes. These functions can also be treated as *the shape functions* resulting from the periodic discretization of the cell using for example the finite element method. The choice of these functions can be also based on the experience or intuition of the researcher.

- The tolerance models can be formulated on the various levels of accuracy. This accuracy is determined by numbers $n \geq 1$ and $N \geq 1$ of terms in the finite sums $h^a(x)U^a_\alpha(x, \xi, t)$ and $g^A(x)W^A(x, \xi, t)$, $x \in \Omega$, $(\xi, t) \in \Xi \times \mathrm{I}$, respectively, occurring in micro-macro decomposition (5.1) and representing micro-fluctuations of displacements caused by a tolerance-periodic structure of the shells under consideration. By increasing the number of n and N we can obtain more detailed descriptions of the investigated problems. However, in the most cases, restriction of considerations to the first terms in series $h^aU^a_\alpha$ and g^AW^A, $a = 1, 2, \ldots, n$, $A = 1, 2, \ldots, N$, i.e. for $a = n = 1$, $A = N = 1$, is sufficient from the calculative point of view, cf. Tomczyk [133], Jędrysiak [27], where some special dynamic problems of thin micro-periodic cylindrical shells [133] and plates [27] are investigated in the framework of the tolerance models. Note, that the micro-periodic shells are special cases of the tolerance-periodic shells considered in this dissertation.

- The resulting equations involve terms with time and spatial derivatives of the fluctuation amplitudes. Hence, these equations describe certain *time-boundary-layer and space-boundary-layer phenomena* strictly related to the specific form of initial and boundary conditions imposed on unknown fluctuation amplitudes U^a_α, W^A, $a = 1, 2, ..., n$, $A = 1, 2, ..., N$.

- It has to be emphasized that solutions to selected initial/boundary value problems formulated in the framework of the tolerance model have a physical sense only if conditions (5.2) hold for the pertinent tolerance parameters δ, i.e. if unknown macrodisplacements u^0_α, w^0 and fluctuation amplitudes U^a_α, W^A of the tolerance model equations are *slowly-varying functions* in the tolerant periodicity direction. These conditions can be also used for the *a posteriori* evaluation of tolerance parameters δ and hence, for the verification of the physical reliability of the obtained solutions.

- For a homogeneous shell with a constant thickness, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $x \in \Omega$, are constant and because $\langle \mu h^a \rangle = \langle \mu g^A \rangle = 0$, $a = 1, 2, ..., n$, $A = 1, 2, ..., N$, we obtain $\langle h^a \rangle = \langle g^A \rangle = 0$, and hence $\langle \partial_1 h^a \rangle = \langle \partial_1 g^A \rangle = \langle \partial_{11} g^A \rangle = 0$. In this case equations $(5.7)_{1,2}$ reduce to the well known shell equations of motion for averaged displacements $u^0_\alpha(x, \xi, t)$, $w^0(x, \xi, t)$ and independently for fluctuation amplitudes $U^a_\alpha(x, \xi, t)$, $W^A(x, \xi, t)$ we arrive at the system of equations, which under condition $\langle f^\beta h^a \rangle = \langle f g^A \rangle = 0$ and under homogeneous initial conditions for U^a_α and W^A, has only trivial solution $U^a_\alpha = W^A = 0$. Hence, from decomposition (5.1) it follows that $u_\alpha = u^0_\alpha$, $w = w^0$. It means that equations (5.6), (5.7) generated by tolerance-averaged Lagrange function (5.3) reduce to the starting equations (4.10) generated by Lagrange function (4.8).

- The tolerance model equations presented here are *more general* than those formulated and discussed in Tomczyk and Szczerba [146], because they are derived without the extra assumption $1 + \lambda/r \approx 1$, where λ and r stand respectively for the microstructure length parameter and the midsurface curvature radius. It means that in the model equations presented here the terms of an order $O(\lambda/r)$ are not neglected.

5.2. Consistent asymptotic model

In this subsection a *new mathematical averaged asymptotic model for the analysis of selected dynamic problems for thin cylindrical shells with a tolerance-periodic microstructure and a functionally (transversally) graded macrostructure in the circumferential direction* will be formulated applying *the consistent asymptotic procedure* proposed in Woźniak *et al.* (eds.) [164].

On passing from the tolerance averaging to the asymptotic averaging, we retain only the concepts of fluctuation shape functions and averaging operation. The notions of slowly-varying and tolerance-periodic functions will not be introduced.

Asymptotic modelling procedure for Euler-Lagrange equations (4.9) is realized in two steps.

The first step is *the consistent asymptotic averaging of lagrangian L defined by* (4.8). To this end we shall restrict considerations to displacement fields $u_\alpha = u_\alpha(x,z,\xi,t)$, $w = w(x,z,\xi,t)$ defined in $\Delta(x)\times\Xi\times\mathrm{I}$, $z\in\Delta(x)$, $x\in\Omega_\Delta$, $(\xi,t)\in\Xi\times\mathrm{I}$. Then, we replace $u_\alpha(x,z,\xi,t)$, $w(x,z,\xi,t)$ by families of displacements $u_{\varepsilon\alpha}(x,z,\xi,t)\equiv u_\alpha\big(x,z/\varepsilon,\xi,t\big)$, $w_\varepsilon(x,z,\xi,t)\equiv w\big(x,z/\varepsilon,\xi,t\big)$, where $\varepsilon\in(0,1]$, $z\in\Delta_\varepsilon(x)$, $\Delta_\varepsilon\equiv(-\varepsilon\lambda/2,\varepsilon\lambda/2)$ (*scaled cell*), $\Delta_\varepsilon(x)\equiv x+\Delta_\varepsilon$, $x\in\Omega_{\Delta_\varepsilon}$ (*scaled cell with a centre at* $x\in\Omega_{\Delta_\varepsilon}$), $\Omega_{\Delta_\varepsilon}\equiv\{x\in\Omega:\Delta_\varepsilon(x)\subset\Omega\}$.

We introduce ***the consistent asymptotic decomposition*** of families of displacements $u_{\varepsilon\alpha}(x,z,\xi,t)$, $w_\varepsilon(x,z,\xi,t)$, $(z,\xi,t)\in\Delta_\varepsilon\times\Xi\times\mathrm{I}$, $x\in\Omega_{\Delta_\varepsilon}$

$$\begin{aligned}
u_{\varepsilon\alpha}(x,z,\xi,t) &\equiv u_\alpha(x,z/\varepsilon,\xi,t) = u^0_\alpha(z,\xi,t)+\varepsilon\widetilde{h}^a_\varepsilon(x,z)U^a_\alpha(z,\xi,t),\\
&\qquad a=1,2,\ldots,n,\\
w_\varepsilon(x,z,\xi,t) &\equiv w(x,z/\varepsilon,\xi,t) = w^0(z,\xi,t)+\varepsilon^2\widetilde{g}^A_\varepsilon(x,z)W^A(z,\xi,t),\\
&\qquad A=1,2,\ldots,N,
\end{aligned}\tag{5.8}$$

where summation convention over a and A holds.

Unknown functions u^0_α, U^a_α in (5.8) are assumed to be continuous and bounded in $\overline{\Omega}$ together with their first derivatives.

Unknown functions w^0, W^A in (5.8) are assumed to be continuous and bounded in $\overline{\Omega}$ together with their derivatives up to the second order.

As in the tolerance modelling, functions u^0_α, w^0 and U^a_α, W^A are called respectively *macrodisplacements* and *fluctuation amplitudes*. We recall that they are not referred to the slowly-varying functions introduced in the tolerance averaging. Moreover, u^0_α, U^a_α, w^0, W^A are assumed to be independent of ε. This is the main difference between the asymptotic approach under consideration and approach which is used in the known homogenization theory, cf. Bensoussan *et al.* [9]; Jikov *et al.* [42].

By $\widetilde{h}^a_\varepsilon(x,z)\equiv\widetilde{h}^a(x,z/\varepsilon)$ and $\widetilde{g}^A_\varepsilon(x,z)\equiv\widetilde{g}^A(x,z/\varepsilon)$, $z\in\Delta_\varepsilon(x)$, $x\in\Omega_{\Delta_\varepsilon}$ in (5.8) are denoted periodic approximations of highly oscillating *fluctuation shape functions* $h^a\in FS^1_\delta(\Omega,\Delta)$ and $g^A\in FS^2_\delta(\Omega,\Delta)$ in $\Delta_\varepsilon(x)$. The *fluctuation shape functions* are assumed to be known in every problem under consideration. They have to satisfy conditions: $h^a_\varepsilon\in O(\varepsilon\lambda)$, $\lambda\partial_1 h^a_\varepsilon\in O(\varepsilon\lambda)$, $g^A_\varepsilon\in O((\varepsilon\lambda)^2)$, $\lambda\partial_1 g^A_\varepsilon\in O((\varepsilon\lambda)^2)$, $\lambda^2\partial_{11}g^A_\varepsilon\in O((\varepsilon\lambda)^2)$, $\langle\mu h^a_\varepsilon\rangle=\langle\mu g^A_\varepsilon\rangle=0$ and $\langle\mu h^a_\varepsilon h^b_\varepsilon\rangle=\langle\mu g^A_\varepsilon g^B_\varepsilon\rangle=0$ for $a\neq b$, $A\neq B$.

Taking into account that $\widetilde{h}^a_\varepsilon(x,z)\equiv\widetilde{h}^a(x,z/\varepsilon)$, $\widetilde{g}^A_\varepsilon(x,z)\equiv\widetilde{g}^A(x,z/\varepsilon)$ and setting $\partial_1\widetilde{h}^a_\varepsilon(x,z)\equiv\varepsilon^{-1}\,\partial_1\widetilde{h}^a(x,z/\varepsilon)$, $\partial_1\widetilde{g}^A_\varepsilon(x,z)\equiv\varepsilon^{-1}\,\partial_1\widetilde{g}^A(x,z/\varepsilon)$, $\partial_{11}\widetilde{g}^A_\varepsilon(x,z)\equiv\varepsilon^{-2}\,\partial_{11}\widetilde{g}^A(x,z/\varepsilon)$, $z\in\Delta_\varepsilon(x)$, $x\in\Omega_{\Delta_\varepsilon}$, from (5.8) we obtain

$$
\begin{aligned}
&u_{\varepsilon\alpha}(x,z,\xi,t)=u^0_\alpha(z,\xi,t)+\varepsilon\widetilde{h}^a(x,z/\varepsilon)U^a_\alpha(z,\xi,t)=u^0_\alpha(z,\xi,t)+O(\varepsilon),\\
&\partial_1u_{\varepsilon\alpha}(x,z,\xi,t)=\partial_1u^0_\alpha(z,\xi,t)+\partial_1\widetilde{h}^a(x,z/\varepsilon)U^a_\alpha(z,\xi,t)+\\
&\quad+\varepsilon\widetilde{h}^a(x,z/\varepsilon)\partial_1U^a_\alpha(z,\xi,t)=\partial_1u^0_\alpha(z,\xi,t)+\\
&\qquad+\partial_1\widetilde{h}^a(x,z/\varepsilon)U^a_\alpha(z,\xi,t)+O(\varepsilon),\\
&\partial_2u_{\varepsilon\alpha}(x,z,\xi,t)=\partial_2u^0_\alpha(z,\xi,t)+\varepsilon\widetilde{h}^a(x,z/\varepsilon)\partial_2U^a_\alpha(z,\xi,t)=\\
&\qquad=\partial_2u^0_\alpha(z,\xi,t)+O(\varepsilon),\\
&\dot{u}_{\varepsilon\alpha}(x,z,\xi,t)=\dot{u}^0_\alpha(z,\xi,t)+\varepsilon\widetilde{h}^a(x,z/\varepsilon)\dot{U}^a_\alpha(z,\xi,t)=\dot{u}^0_\alpha(z,\xi,t)+O(\varepsilon),\\
&w_\varepsilon(x,z,\xi,t)=w^0(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)W^A(z,\xi,t)=w^0(z,\xi,t)+O(\varepsilon^2),\\
&\partial_1w_\varepsilon(x,z,\xi,t)=\partial_1w^0(z,\xi,t)+\varepsilon\partial_1\widetilde{g}^A(x,z/\varepsilon)W^A(z,\xi,t)+\\
&\qquad+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\partial_1W^A(z,\xi,t)=\partial_1w^0(z,\xi,t)+O(\varepsilon)+O(\varepsilon^2),\\
&\partial_{11}w_\varepsilon(x,z,\xi,t)=\partial_{11}w^0(z,\xi,t)+\partial_{11}\widetilde{g}^A(x,z/\varepsilon)W^A(z,\xi,t)+\\
&\qquad+2\varepsilon\partial_1\widetilde{g}^A(x,z/\varepsilon)W^A(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\partial_{11}W^A(z,\xi,t)=\\
&\qquad=\partial_{11}w^0(z,\xi,t)+\partial_{11}\widetilde{g}^A(x,z/\varepsilon)W^A(z,\xi,t)+O(\varepsilon)+O(\varepsilon^2),\\
&\partial_{12}w_\varepsilon(x,z,\xi,t)=\partial_{21}w_\varepsilon(x,z,\xi,t)=\partial_{12}w^0(z,\xi,t)+\\
&\qquad+\varepsilon\partial_1\widetilde{g}^A(x,z/\varepsilon)\partial_2W^A(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\partial_{12}W^A(z,\xi,t)=\\
&\qquad=\partial_{12}w^0(z,\xi,t)+O(\varepsilon)+O(\varepsilon^2),\\
&\partial_2w_\varepsilon(x,z,\xi,t)=\partial_2w^0(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\partial_2W^A(z,\xi,t)=\\
&\qquad=\partial_2w^0(z,\xi,t)+O(\varepsilon^2),\\
&\partial_{22}w_\varepsilon(x,z,\xi,t)=\partial_{22}w^0(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\partial_{22}W^A(z,\xi,t)=\\
&\qquad=\partial_{22}w^0(z,\xi,t)+O(\varepsilon^2),\\
&\dot{w}_\varepsilon(x,z,\xi,t)=\dot{w}^0(z,\xi,t)+\varepsilon^2\widetilde{g}^A(x,z/\varepsilon)\dot{W}^A(z,\xi,t)=\dot{w}^0(z,\xi,t)+O(\varepsilon^2),\\
&z\in\Delta_\varepsilon(x),\quad x\in\Omega_{\Delta_\varepsilon},\quad (\xi,t)\in\Xi\times\mathrm{I}.
\end{aligned}
\tag{5.9}
$$

Due to the fact that lagrangian L defined by (4.8) is highly oscillating with respect to x, there exists for every $x\in\Omega_\Delta$ lagrangian $\widetilde{L}\left(x,z,\xi,t,\partial_\beta u_\alpha,u_\alpha,\dot{u}_\alpha,\partial_{\alpha\beta}w,w,\dot{w}\right)$ which constitutes a periodic approximation of lagrangian L in $\Delta(x)$, $z\in\Delta(x)$, $x\in\Omega_\Delta$. Let $\widetilde{L}_\varepsilon$ be a family of functions given by

$$
\begin{aligned}
\widetilde{L}_\varepsilon&=\widetilde{L}\left(x,z/\varepsilon,\xi,t,\partial_\beta u_{\varepsilon\alpha},u_{\varepsilon\alpha},\dot{u}_{\varepsilon\alpha},\partial_{\alpha\beta}w_\varepsilon,w_\varepsilon,\dot{w}_\varepsilon\right)=\\
&=-\frac{1}{2}\Big[\widetilde{D}^{\alpha\beta\gamma\delta}\partial_\beta u_{\varepsilon\alpha}\partial_\delta u_{\varepsilon\gamma}+2r^{-1}\widetilde{D}^{\alpha\beta11}w_\varepsilon\partial_\beta u_{\varepsilon\alpha}+r^{-2}\widetilde{D}^{1111}(w_\varepsilon)^2+\\
&+\widetilde{B}^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_\varepsilon\partial_{\gamma\delta}w_\varepsilon-\widetilde{\mu}a^{\alpha\beta}\dot{u}_{\varepsilon\alpha}\dot{u}_{\varepsilon\beta}-\widetilde{\mu}(\dot{w}_\varepsilon)^2\Big]+f^\alpha u_{\varepsilon\alpha}+fw_\varepsilon,
\end{aligned}
\tag{5.10}
$$

where $\widetilde{D}^{\alpha\beta\gamma\delta}$, $\widetilde{B}^{\alpha\beta\gamma\delta}$, $\widetilde{\mu}$ are periodic approximation of $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$, μ respectively.

Substituting the right-hand sides of (5.9) into (5.10) and taking into account that under limit passage $\varepsilon \to 0$ for $z \in \Delta_\varepsilon(x)$, terms $O(\varepsilon)$, $O(\varepsilon^2)$ (i.e. terms depending on ε and ε^2) can be neglected as well as bearing in mind that if $\varepsilon \to 0$ then every continuous and bounded function $p(z,\xi,t)$, $z \in \Delta_\varepsilon(x)$, $x \in \Omega_{\Delta_\varepsilon}$, $(\xi,t) \in \Xi \times \mathrm{I}$, tends to function $p(x,\xi,t)$, $x \in \overline{\Omega}$, $(\xi,t) \in \Xi \times \mathrm{I}$, we arrive at

$$\begin{aligned} \widetilde{L}_\varepsilon =& \widetilde{L}\Big(x, z/\varepsilon, \xi, t, \partial_1 u^0_\alpha(x,\xi,t) + \partial_1 \widetilde{h}^a(x,z/\varepsilon) U^a_\alpha(x,\xi,t), \partial_2 u^0_\alpha(x,\xi,t), \\ & u^0_\alpha(x,\xi,t), \dot{u}^0_\alpha(x,\xi,t), \partial_{11} w^0(x,\xi,t) + \partial_{11}\widetilde{g}^A(x,z/\varepsilon) W^A(x,\xi,t), \\ & \partial_{12} w^0(x,\xi,t), \partial_{21} w^0(x,\xi,t), \partial_{22} w^0(x,\xi,t), w^0(x,\xi,t), \dot{w}^0(x,\xi,t)\Big). \end{aligned} \tag{5.11}$$

Moreover, if $\varepsilon \to 0$ then, by means of a property of the mean value, cf. Jikov *et al.* [42], the obtained result tends weakly to $L_0\left(x, \partial_\beta u^0_\alpha, u^0_\alpha, U^a_\alpha, \dot{u}^0_\alpha, \partial_{\alpha\beta} w^0, w^0, W^A, \dot{w}^0\right)$, where

$$\begin{aligned} L_0 =& \Big\langle \widetilde{L}\Big(x, z, \xi, t, \partial_1 u^0_\alpha(x,\xi,t) + \partial_1 \widetilde{h}^a(x,z) U^a_\alpha(x,\xi,t), \partial_2 u^0_\alpha(x,\xi,t), u^0_\alpha(x,\xi,t), \\ & \dot{u}^0_\alpha(x,\xi,t), \partial_{11} w^0(x,\xi,t) + \partial_{11}\widetilde{g}^A(x,z) W^A(x,\xi,t), \partial_{12} w^0(x,\xi,t), \\ & \partial_{21} w^0(x,\xi,t), \partial_{22} w^0(x,\xi,t), w^0(x,\xi,t), \dot{w}^0(x,\xi,t)\Big)\Big\rangle. \end{aligned}$$

The explicit form of L_0 is given by

$$\begin{aligned} & L_0\left(x, \partial_\beta u^0_\alpha, u^0_\alpha, U^a_\alpha, \dot{u}^0_\alpha, \partial_{\alpha\beta} w^0, w^0, W^A, \dot{w}^0\right) = \\ & = -\frac{1}{2}\Big[\left\langle D^{\alpha\beta\gamma\delta}\right\rangle \partial_\beta u^0_\alpha \partial_\delta u^0_\gamma + 2\left\langle D^{\alpha\beta\gamma 1}\partial_1 h^a\right\rangle \partial_\beta u^0_\alpha U^a_\gamma + \\ & + \left\langle D^{\alpha 1\gamma 1}\partial_1 h^a \partial_1 h^b\right\rangle U^a_\gamma U^b_\alpha + 2r^{-1}\Big(\left\langle D^{\alpha\beta 11}\right\rangle \partial_\beta u^0_\alpha w^0 + \\ & + \left\langle D^{\alpha 111}\partial_1 h^a\right\rangle w^0 U^a_\alpha\Big) + r^{-2}\left\langle D^{1111}\right\rangle \left(w^0\right)^2 + \\ & + \left\langle B^{\alpha\beta\gamma\delta}\right\rangle \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 + 2\left\langle B^{\alpha\beta 11}\partial_{11} g^A\right\rangle \partial_{\alpha\beta} w^0 W^A + \\ & + \left\langle B^{1111}\partial_{11} g^A \partial_{11} g^B\right\rangle W^A W^B - \langle\mu\rangle a^{\alpha\beta}\dot{u}^0_\alpha \dot{u}^0_\beta + \\ & - \langle\mu\rangle \left(\dot{w}^0\right)^2\Big] + \langle f^\alpha\rangle u^0_\alpha + \langle f\rangle w^0, \end{aligned} \tag{5.12}$$

where averages $\langle\cdot\rangle$ on the right-hand side of (5.12) are continuous and slowly-varying in x and calculated by means of (3.5).

Function L_0, given above, is ***the averaged form of lagrangian L defined by (4.8) under consistent asymptotic averaging***. We recall that concept of

L_0 was introduced without any reference to the concept of slowly-varying and tolerance-periodic functions.

In the framework of consistent asymptotic modelling we introduce ***the consistent asymptotic action functional*** defined by

$$A^0_{hg}\left(u^0_\alpha, U^a_\alpha, w^0, W^A\right) = \int\limits_0^{L_1}\int\limits_0^{L_2}\int\limits_{t_0}^{t_1} L_0 dt d\xi dx, \tag{5.13}$$

where L_0 is given by (5.12).

Under assumption that $\partial L_0/\partial_\beta u^0_\alpha$, $\partial L_0/\partial_{\alpha\beta} w^0$ are continuous, from the principle of stationary action applied to (5.13), we obtain the following system of Euler-Lagrange equations

$$\begin{aligned}
&\partial_\beta \frac{\partial L_0}{\partial\left(\partial_\beta u^0_\alpha\right)} - \frac{\partial L_0}{\partial u^0_\alpha} + \frac{\partial}{\partial t}\frac{\partial L_0}{\partial \dot{u}^0_\alpha} = 0,\\
&-\partial_{\alpha\beta} \frac{\partial L_0}{\partial\left(\partial_{\alpha\beta} w^0\right)} - \frac{\partial L_0}{\partial w^0} + \frac{\partial}{\partial t}\frac{\partial L_0}{\partial \dot{w}^0} = 0,\\
&\frac{\partial L_0}{\partial U^a_\alpha} = 0, \quad a = 1, 2, \ldots, n,\\
&\frac{\partial L_0}{\partial W^A} = 0, \quad A = 1, 2, \ldots, N.
\end{aligned} \tag{5.14}$$

Combining (5.14) with (5.12) we arrive at the explicit form of *the consistent asymptotic model equations* for $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$, $U^a_\alpha(x,\xi,t)$, $W^A(x,\xi,t)$, $x \in \Omega$, $(\xi, t) \in \Xi \times \mathrm{I}$

$$\begin{aligned}
&\partial_\beta\left(\left\langle D^{\alpha\beta\gamma\delta}\right\rangle \partial_\delta u^0_\gamma + r^{-1}\left\langle D^{\alpha\beta 11}\right\rangle w^0\right) + \left\langle D^{\alpha\beta\gamma 1}\partial_1 h^b\right\rangle \partial_\beta U^b_\gamma +\\
&-\langle\mu\rangle a^{\alpha\beta}\ddot{u}^0_\beta + \langle f^\alpha\rangle = 0,\\
&\partial_{\alpha\beta}\left(\left\langle B^{\alpha\beta\gamma\delta}\right\rangle \partial_{\gamma\delta} w^0 + \left\langle D^{\alpha\beta 11}\partial_{11} g^B\right\rangle W^B\right) + r^{-1}\left\langle D^{11\gamma\delta}\right\rangle \partial_\delta u^0_\gamma +\\
&+ r^{-2}\left\langle D^{1111}\right\rangle w^0 + r^{-1}\left\langle D^{111\delta}\partial_1 h^b\right\rangle U^b_\delta + \langle\mu\rangle \ddot{w}^0 - \langle f\rangle = 0,\\
&\left\langle \partial_1 h^a D^{1\beta\gamma 1}\partial_1 h^b\right\rangle U^b_\gamma = -\left\langle \partial_1 h^a D^{\beta 1\gamma\delta}\right\rangle \partial_\delta u^0_\gamma - r^{-1}\left\langle \partial_1 h^a D^{1\beta 11}\right\rangle w^0,\\
&\left\langle \partial_{11} g^A B^{1111}\partial_{11} g^B\right\rangle W^A = -\left\langle \partial_{11} g^B B^{11\gamma\delta}\right\rangle \partial_{\gamma\delta} w^0,\\
&a, b = 1, 2, \ldots, n, \quad A, B = 1, 2, \ldots, N.
\end{aligned} \tag{5.15}$$

Equations (5.15) consist of three partial differential equations for macrodisplacements u^0_α, w^0 coupled with $(2n+N)$ linear algebraic equations for fluctuation amplitudes U^a_α, W^A.

It can be shown that linear transformations $\mathbf{G}$, $\mathbf{E}$ given by $G^{ab}_{\alpha\gamma} = \langle \partial_1 h^a D^{\alpha 1 \gamma 1} \partial_1 h^b \rangle$, $E^{AB} = \langle \partial_{11} g^A B^{1111} \partial_{11} g^B \rangle$, respectively, are invertible. Hence, solutions U^b_γ, W^A to equations $(5.15)_{3,4}$ can be written in the form

$$
\begin{aligned}
U^b_\gamma &= -\left(G^{-1}\right)^{bc}_{\gamma\eta} \left[\left\langle \partial_1 h^c D^{1\eta\mu\vartheta} \right\rangle \partial_\vartheta u^0_\mu + r^{-1} \left\langle \partial_1 h^c D^{1\eta 11} \right\rangle w^0 \right], \\
W^A &= -\left(E^{-1}\right)^{AB} \left\langle \partial_{11} g^B B^{11\gamma\delta} \right\rangle \partial_{\gamma\delta} w^0,
\end{aligned}
\tag{5.16}
$$

where $\mathbf{G}^{-1}$ and $\mathbf{E}^{-1}$ are the inverses of the linear transformations $\mathbf{G}$, $\mathbf{E}$ respectively. Substituting (5.16) into $(5.15)_{1,2}$ and setting

$$
\begin{aligned}
D^{\alpha\beta\gamma\delta}_h &\equiv \left\langle D^{\alpha\beta\gamma\delta} \right\rangle - \left\langle D^{\alpha\beta\eta 1} \partial_1 h^a \right\rangle \left(G^{-1}\right)^{ab}_{\eta\zeta} \left\langle \partial_1 h^b D^{1\zeta\gamma\delta} \right\rangle, \\
B^{\alpha\beta\gamma\delta}_g &\equiv \left\langle B^{\alpha\beta\gamma\delta} \right\rangle - \left\langle B^{\alpha\beta 11} \partial_{11} g^A \right\rangle \left(E^{-1}\right)^{AB} \left\langle \partial_{11} g^B B^{11\gamma\delta} \right\rangle,
\end{aligned}
\tag{5.17}
$$

we arrive finally at the following form of Euler-Lagrange equations for $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$, $x \in \Omega$, $(\xi,t) \in \Xi \times \mathrm{I}$,

$$
\begin{aligned}
&\partial_\beta \left(D^{\alpha\beta\gamma\delta}_h \partial_\delta u^0_\gamma + r^{-1} D^{\alpha\beta 11}_h w^0 \right) - \langle \mu \rangle a^{\alpha\beta} \ddot{u}^0_\beta + \langle f^\alpha \rangle = 0, \\
&\partial_{\alpha\beta} \left(B^{\alpha\beta\gamma\delta}_g \partial_{\gamma\delta} w^0 \right) + r^{-1} D^{11\gamma\delta}_h \partial_\delta u^0_\gamma + r^{-2} D^{1111}_h w^0 + \langle \mu \rangle \ddot{w}^0 - \langle f \rangle = 0.
\end{aligned}
\tag{5.18}
$$

Since functions $u_\alpha(x,\xi,t)$, $w(x,\xi,t)$ have to be uniquely defined in $\Omega \times \Xi \times \mathrm{I}$, we conclude that $u_\alpha(x,\xi,t)$, $w(x,\xi,t)$ have to take the form

$$
\begin{aligned}
u_\alpha(x,\xi,t) &= u^0_\alpha(x,\xi,t) + h^a(x) U^a_\alpha(x,\xi,t), \\
w(x,\xi,t) &= w^0(x,\xi,t) + g^A(x) W^A(x,\xi,t), \quad x \in \Omega, (\xi,t) \in \Xi \times \mathrm{I},
\end{aligned}
\tag{5.19}
$$

with U^a_α, W^A given by (5.16). Contrary to (5.1), the above formula is not a micro-macro decomposition since in the consistent asymptotic approach it is not assumed that functions u^0_α, w^0, U^a_α, W^A are slowly-varying. Tensors $D^{\alpha\beta\gamma\delta}_h$, $B^{\alpha\beta\gamma\delta}_g$ given by (5.17) are *tensors of effective elastic moduli* for the shells under consideration.

Equations (5.18) for macrodisplacements u^0_α, w^0 together with formula (5.19) and with expressions (5.16) for fluctuation amplitudes U^a_α, W^A represent ***the consistent asymptotic model of selected dynamic problems for the thin transversally graded cylindrical shells under consideration.***

Discussion of results

The characteristic features of the derived consistent asymptotic model are:

- In contrast to starting equations (4.10) with discontinuous, highly oscillating and tolerance-periodic coefficients, the asymptotic model equations (5.18) proposed here ***have continuously slowly-varying coefficients***.

- Contrary to tolerance model equations (5.6) and (5.7), *the asymptotic model is not able to describe the length-scale effect on the overall shell dynamics* being independent of microstructure cell size λ.

- The number and form of boundary/initial conditions for unknowns u_α^0, w^0 are the same as in the classical shell theory governed by equations (4.10).

- The extra unknown functions called *fluctuation amplitudes* U_α^a, W^A are governed by the system of $(2n + N)$ linear algebraic equations $(5.15)_{3,4}$ and can be always eliminated from the system of governing equations (5.15) by means of (5.16). Hence, the unknowns of final asymptotic model equations (5.18) are only macrodisplacements u_α^0, w^0.

- The resulting asymptotic equations (5.18) are uniquely determined by the postulated *a priori* tolerance-periodic *fluctuations shape functions*, $h^a \in FS_\delta^1(\Omega, \Delta)$, $h^a \in O(\lambda)$, $a = 1, 2, \ldots, n$ and $g^A \in FS_\delta^2(\Omega, \Delta)$, $g^A \in O(\lambda^2)$, $A = 1, 2, \ldots, N$ representing oscillations inside a cell. These functions can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell. These functions can also be derived by means of the periodic discretization of the cell using for example the finite element method. The choice of these functions can be also based on the experience or intuition of the researcher.

- If the fluctuation shape functions are not derived as solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell then ***the effective moduli (5.17) of the shell are obtained without specification of the periodic cell problems***. This situation is different from that occurring in the known asymptotic homogenisation approach, cf. e.g. Bensoussan *et al.* [9], where ***only solutions to the periodic cell problems make it possible to define the effective moduli*** of the structure under consideration

- For a homogeneous shell with a constant thickness, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$ are constant and because $\langle \mu h^a \rangle = \langle \mu g^A \rangle = 0$, we obtain $\langle h^a \rangle = \langle g^A \rangle = 0$, and hence $\langle \partial_1 h^a \rangle = \langle \partial_1 g^A \rangle = \langle \partial_{11} g^A \rangle = 0$. In this case we obtain from (5.16) that

$U^a_\alpha = W^A = 0$ and from (5.17) that $D_h^{\alpha\beta\gamma\delta} \equiv D^{\alpha\beta\gamma\delta}$, $B_g^{\alpha\beta\gamma\delta} \equiv B^{\alpha\beta\gamma\delta}$. Thus, from decomposition (5.19) it follows that $u_\alpha = u^0_\alpha$, $w = w^0$. It means that equations (5.18) generated by asymptotically averaged Lagrange function (5.12) reduce to the starting equations (4.10) generated by Lagrange function (4.8).

- It may also be noticed that from the formal point of view, asymptotic model equations (5.18) can be obtained directly from tolerance model equations (5.6) and (5.7) by the formal limit passage $\lambda \to 0$, i.e. after neglecting terms depending on microstructure length parameter λ (underlined terms).

5.3. Combined asymptotic-tolerance model

In this subsection *a new mathematical averaged asymptotic-tolerance model for the analysis of selected dynamic problems for thin cylindrical shells with a tolerance-periodic microstructure and a functionally (transversally) graded macrostructure in the circumferential direction* will be formulated by applying *the combined modelling* proposed in Woźniak *et al.* (eds.) [164]. This combined modelling includes both the consistent asymptotic and the tolerance non-asymptotic modelling techniques which are combined together into a new procedure.

The combined modelling technique is realized in two steps.

Step 1. Consistent asymptotic modelling

The first step is based on ***the consistent asymptotic procedure*** which leads from starting equations (4.9) with highly oscillating and discontinuous coefficients to the Euler-Lagrange equations with *continuous and slowly-varying coefficients* independent of the microstructure cell size. Hence the model obtained in the first step is referred to as ***the macroscopic model***. This consistent asymptotic (macroscopic) model has been formulated in Subsection 5.2 and consists of equations (5.18) and decomposition (5.19) in which the fluctuation amplitudes are given by (5.16).

In the subsequent considerations, external forces f^a, f will be neglected.

Below, we rewrite equations (5.18) of ***the consistent asymptotic (macroscopic) model*** without the external forces

$$\begin{aligned}
&\partial_\beta \left(D_h^{\alpha\beta\gamma\delta} \partial_\delta u^0_\gamma + r^{-1} D_h^{\alpha\beta 11} w^0 \right) - \langle \mu \rangle \, a^{\alpha\beta} \ddot{u}^0_\beta = 0 \\
&\partial_{\alpha\beta} \left(B_g^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} w^0 \right) + r^{-1} D_h^{11\gamma\delta} \partial_\delta u^0_\gamma + r^{-2} D_h^{1111} w^0 + \langle \mu \rangle \, \ddot{w}^0 = 0.
\end{aligned} \tag{5.20}$$

In the first step of combined modelling it is assumed that functions u^0_α, w^0 obtained as solution to a certain boundary-initial value problem for consistent asymptotic equations (5.20) are known. Hence, there are also known functions

$$\begin{aligned} &u_{0\alpha}(x,\xi,t) = u^0_\alpha(x,\xi,t) + h^a(x)U^a_\alpha(x,\xi,t),\\ &w_0(x,\xi,t) = w^0(x,\xi,t) + g^A(x)W^A(x,\xi,t),\\ &x\in\Omega,\quad (\xi,t)\in\Xi\times\mathrm{I},\quad a=1,2,\ldots,n,\quad A=1,2,\ldots,N, \end{aligned} \tag{5.21}$$

where U^a_α, W^A are given by means of (5.16).

Step 2. Tolerance modelling

The second step of the combined modelling will be realized by means of ***the tolerance (non-asymptotic) modelling procedure***.

In the second step we introduce the ***new*** tolerance-periodic in $x \in \Omega$, continuous and highly-oscillating ***fluctuation shape functions***: $c^k \in FS^1_\delta(\Omega,\Delta)$, $k = 1,2,\ldots,m$, $b^K \in FS^2_\delta(\Omega,\Delta)$, $K = 1,2,\ldots,M$, such that $c^k \in O(\lambda)$, $\lambda\partial_1 c^k \in O(\lambda)$, $b^K \in O(\lambda^2)$, $\lambda\partial_1 b^K \in O(\lambda^2)$, $\lambda^2\partial_{11} b^K \in O(\lambda^2)$, $\langle\mu c^k\rangle = \langle\mu b^K\rangle = 0$ and $\langle\mu c^k c^l\rangle = \langle\mu b^K b^L\rangle = 0$ for $k \neq l$, $K \neq L$, where $\mu(x)$ is the shell mass density being a tolerance-periodic function with respect to x. These functions are assumed to be known in every problem under consideration. Taking into account that inside the cell the geometrical, elastic and inertial properties of the shells under consideration are described by symmetric (i.e. even) functions of argument $z \in \Delta(x)$ (the cell has a symmetry axis for $z = 0$), we assume that periodic approximations $\widetilde{c}^k(x,z)$ and $\widetilde{b}^K(x,z)$ of $c^k(x)$ and $b^K(x)$ in $\Delta(x)$, $x \in \Omega_\Delta$, are either even or odd functions with respect to z.

Let functions $Q^k_\alpha(x,\xi,t)$, $k = 1,2,\ldots,m$ and $V^K(x,\xi,t)$, $K = 1,2,\ldots,M$, $(x,\xi,t) \in \Omega\times\Xi\times\mathrm{I}$, be ***the new unknowns called fluctuation (microscopic) amplitudes***, which are slowly-varying in x, $Q^k_\alpha \in SV^1_\delta(\Omega,\Delta)$, $V^K \in SV^2_\delta(\Omega,\Delta)$.

We shall introduce ***the extra micro-macro decomposition*** *superimposed on the known solutions* $u_{0\alpha} \in TP^1_\delta(\Omega,\Delta)$, $w_0 \in TP^2_\delta(\Omega,\Delta)$ obtained within the macroscopic model

$$\begin{aligned} &u_{c\alpha}(x,\xi,t) = u_{0\alpha}(x,\xi,t) + c^k(x)Q^k_\alpha(x,\xi,t),\\ &w_b(x,\xi,t) = w_0(x,\xi,t) + b^K(x)V^K(x,\xi,t),\\ &x\in\Omega,\quad (\xi,t)\in\Xi\times\mathrm{I},\quad k=1,2,\ldots,m,\quad K=1,2,\ldots,M, \end{aligned} \tag{5.22}$$

where summation convention over k and K holds and where $u_{c\alpha} \in TP^1_\delta(\Omega,\Delta)$, $w_b \in TP^2_\delta(\Omega,\Delta)$. Formula (5.22) will be also referred to as ***decomposition superimposed on the first step of combined modelling***.

Due to the fact that $u_{c\alpha}(\cdot,\xi,t) \in TP^1_\delta(\Omega,\Delta)$, $w_b(\cdot,\xi,t) \in TP^2_\delta(\Omega,\Delta)$ are tolerance-periodic functions, there exist periodic approximations of these functions and of their pertinent derivatives in every $\Delta(x)$, $x \in \Omega_\Delta$. Bearing in mind the properties of the slowly-varying functions and fluctuation shape functions and taking into account that $u_{0\alpha}(x,\xi,t)$ and $w_0(x,\xi,t)$ given by (5.21) are tolerance-periodic functions in x, i.e. $u_{0\alpha} \in TP^1_\delta(\Omega,\Delta)$, $w_0 \in TP^2_\delta(\Omega,\Delta)$, we arrive at the following results in $\Delta(x) \times \Xi \times \mathrm{I}$

$$\begin{aligned}
&\widetilde{u}_{c\alpha}(x,z,\xi,t) = \widetilde{u}_{0\alpha}(x,z,\xi,t) + \widetilde{c}^k(x,z)Q^k_\alpha(x,\xi,t),\\
&\partial_1\widetilde{u}_{c\alpha}(x,z,\xi,t) = \partial_1\widetilde{u}_{0\alpha}(x,z,\xi,t) + \partial_1\widetilde{c}^k(x,z)Q^k_\alpha(x,\xi,t),\\
&\partial_2\widetilde{u}_{c\alpha}(x,z,\xi,t) = \partial_2\widetilde{u}_{0\alpha}(x,z,\xi,t) + \widetilde{c}^k(x,z)\partial_2 Q^k_\alpha(x,\xi,t),\\
&\dot{\widetilde{u}}_{c\alpha}(x,z,\xi,t) = \dot{\widetilde{u}}_{0\alpha}(x,z,\xi,t) + \widetilde{c}^k(x,z)\dot{Q}^k_\alpha(x,\xi,t),
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
&\widetilde{w}_b(x,z,\xi,t) = \widetilde{w}_0(x,z,\xi,t) + \widetilde{b}^K(x,z)V^K(x,\xi,t),\\
&\partial_1\widetilde{w}_b(x,z,\xi,t) = \partial_1\widetilde{w}_0(x,z,\xi,t) + \partial_1\widetilde{b}^K(x,z)V^K(x,\xi,t),\\
&\partial_{11}\widetilde{w}_b(x,z,\xi,t) = \partial_{11}\widetilde{w}_0(x,z,\xi,t) + \partial_{11}\widetilde{b}^K(x,z)V^K(x,\xi,t),\\
&\partial_{12}\widetilde{w}_b(x,z,\xi,t) = \partial_{21}\widetilde{w}_b(x,z,\xi,t) = \partial_{12}\widetilde{w}_0(x,z,\xi,t)+\\
&\qquad\qquad + \partial_1\widetilde{b}^K(x,z)\partial_2 V^K(x,\xi,t),\\
&\partial_2\widetilde{w}_b(x,z,\xi,t) = \partial_2\widetilde{w}_0(x,z,\xi,t) + \widetilde{b}^K(x,z)\partial_2 V^K(x,\xi,t),\\
&\partial_{22}\widetilde{w}_b(x,z,\xi,t) = \partial_{22}\widetilde{w}_0(x,z,\xi,t) + \widetilde{b}^K(x,z)\partial_{22} V^K(x,\xi,t),\\
&\dot{\widetilde{w}}_b(x,z,\xi,t) = \dot{\widetilde{w}}_0(x,z,\xi,t) + \widetilde{b}^K(x,z)\dot{V}^K(x,\xi,t),
\end{aligned} \tag{5.24}$$

where $z \in \Delta(x)$, $x \in \Omega_\Delta$, $(\xi,t) \in \Xi \times \mathrm{I}$ and $\partial_1\widetilde{c}^k(x,z)$, $\partial_1\widetilde{b}^K(x,z)$, $\partial_{11}\widetilde{b}^K(x,z)$ stand for derivatives of $\widetilde{c}^k(\cdot)$ and $\widetilde{b}^K(\cdot)$ with respect to $z \in \Delta(x)$. Obviously, in terms

$$\partial_1\widetilde{u}_{0\alpha}(x,z,\xi,t) = \partial_1\left(u^0_\alpha(x,\xi,t) + \widetilde{h}^a(x,z)U^a_\alpha(x,\xi,t)\right),$$

$$\partial_1\widetilde{w}_0(x,z,\xi,t) = \partial_1\left(w^0(x,\xi,t) + \widetilde{g}^A(x,z)W^A(x,\xi,t)\right),$$

$$\partial_{11}\widetilde{w}_0(x,z,\xi,t) = \partial_{11}\left(w^0(x,\xi,t) + \widetilde{g}^A(x,z)W^A(x,\xi,t)\right),$$

we deal with derivatives of u^0_α, w^0 with respect to $x \in \Omega$ and with derivatives of $\widetilde{h}^a$, $\widetilde{g}^A$ with respect to $z \in \Delta(x)$.

Setting $u_{c_\alpha} \equiv u_\alpha$, $w_b \equiv w$ and after neglecting the external forces we obtain from (4.8) lagrangian $L_{cb}(x,\xi,t,\partial_\beta u_{c\alpha},\dot{u}_{c\alpha},\partial_{\alpha\beta}w_b,w_b,\dot{w}_b)$, $x \in \Omega$, $(\xi,t) \in \Xi \times \mathrm{I}$, having the following form

$$
\begin{aligned}
L_{cb} = -\frac{1}{2}\Bigg(& D^{\alpha\beta\gamma\delta}\partial_\beta u_{c\alpha}\partial_\delta u_{c\gamma} + \frac{2}{r}D^{\alpha\beta 11}w_b\partial_\beta u_{c\alpha} + \frac{1}{r^2}D^{1111}w_b w_b + \\
& + B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_b\partial_{\gamma\delta}w_b - \mu a^{\alpha\beta}\dot{u}_{c\alpha}\dot{u}_{c\beta} - \mu\left(\dot{w}_b\right)^2\Bigg).
\end{aligned}
\tag{5.25}
$$

Action functional $A(u_{cb}, w_b)$ determined by L_{cb} is defined by

$$
A(u_{c\alpha}, w_b) = \int_0^{L_1}\int_0^{L_2}\int_{t_0}^{t_1} L_{cb}(x,\xi,t,\partial_\beta u_{c\alpha},\dot{u}_{c\alpha},\partial_{\alpha\beta}w_b,w_b,\dot{w}_b)\,dt\,d\xi\,dx. \tag{5.26}
$$

Since lagrangian L_{cb} is highly oscillating with respect to x then there exists a periodic approximation $\widetilde{L}_{cb}(x,z,\xi,t,\partial_\beta\widetilde{u}_{c\alpha},\dot{\widetilde{u}}_{c\alpha},\partial_{\alpha\beta}\widetilde{w}_b,\widetilde{w}_b,\dot{\widetilde{w}}_b)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, $(\xi, t) \in \Omega \times \mathrm{I}$, of L_{cb} in every $\Delta(x)$. Substituting the right hand sides of approximations (5.23), (5.24) into this lagrangian as well as substituting into $\widetilde{L}_{cb}$ the periodic approximations $\widetilde{D}^{\alpha\beta\gamma\delta}(x,z)$, $\widetilde{B}^{\alpha\beta\gamma\delta}(x,z)$, $\widetilde{\mu}(x,z)$ of tolerance-periodic functions $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$, $\mu(x) \in TP^0_\delta(\Omega,\Delta)$ and averaging $\widetilde{L}_{cb}$ over cell $\Delta(x)$ using tolerance averaging formula (3.5) and *tolerance averaging approximation* (3.6), we arrive at function $\langle L_{cb}\rangle$ being *the tolerance averaging of lagrangian* $L_{cb}(x,\xi,t,\partial_\beta u_{c\alpha},\dot{u}_{c\alpha},\partial_{\alpha\beta}w_b,w_b,\dot{w}_b)$ *in* $\Delta(x)$ *under micro-macro decomposition* (5.22). Introducing the extra approximation $1 + \lambda/r \approx 1$, where r is the midsurface curvature radius, as well as recalling that $u_{0\alpha}(\cdot,\xi,t) \in TP^1_\delta(\Omega,\Delta)$ and $w_0(\cdot,\xi,t) \in TP^2_\delta(\Omega,\Delta)$ in (5.22) are known, the obtained result has the form

$$
\begin{aligned}
& \langle L_{cb}\rangle\left(x,\partial_2 Q^k_\alpha, Q^k_\alpha, \dot{Q}^k_\alpha, \partial_{22}V^K, \partial_2 V^K, V^K, \dot{V}^K\right) = \\
& = -\frac{1}{2}\Bigg[\left\langle D^{\alpha\beta\gamma\delta}\partial_\beta u_{0\alpha}\partial_\delta u_{0\gamma}\right\rangle + 2\left\langle D^{\alpha\beta\gamma 1}\partial_1 c^k\partial_\beta u_{0\alpha}\right\rangle Q^k_\gamma + \\
& + \left\langle D^{\alpha 11\gamma}\partial_1 c^k\partial_1 c^l\right\rangle Q^k_\gamma Q^l_\alpha + \underline{\left\langle D^{\alpha 22\gamma}c^k c^l\right\rangle}\partial_2 Q^l_\gamma\partial_2 Q^k_\alpha + \\
& + 2r^{-1}\left(\left\langle D^{\alpha\beta 11}\partial_\beta u_{0\alpha}w_0\right\rangle + \left\langle D^{\alpha 111}\partial_1 c^k w_0\right\rangle Q^k_\alpha\right) + \\
& + r^{-2}\left\langle D^{1111}w_0 w_0\right\rangle + \left\langle B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_0\partial_{\gamma\delta}w_0\right\rangle + \\
& + 2\Bigg(\left\langle B^{\alpha\beta 11}\partial_{11}b^K\partial_{\alpha\beta}w_0\right\rangle V^K + \underline{\left\langle B^{\alpha\beta 22}b^K\partial_{\alpha\beta}w_0\right\rangle}\partial_{22}V^K + \\
& + \underline{\left\langle B^{1122}b^K\partial_{11}b^L\right\rangle}\partial_{22}V^L V^K\Bigg) + 4\underline{\left\langle B^{1212}\partial_1 b^K\partial_1 b^L\right\rangle}\partial_2 V^K\partial_2 V^L + \\
& + \left\langle B^{1111}\partial_{11}b^K\partial_{11}b^L\right\rangle V^K V^L + \underline{\left\langle B^{2222}b^K b^L\right\rangle}\partial_{22}V^K\partial_{22}V^L + \\
& - \left\langle\mu a^{\alpha\beta}\dot{u}_{0\alpha}\dot{u}_{0\beta}\right\rangle - \left\langle\mu\left(\dot{w}_0\right)^2\right\rangle - \underline{\left\langle\mu c^k c^l\right\rangle}a^{\alpha\beta}\dot{Q}^k_\alpha\dot{Q}^l_\beta - \underline{\left\langle\mu b^K b^L\right\rangle}\dot{V}^K\dot{V}^L\Bigg].
\end{aligned}
\tag{5.27}
$$

The underlined terms in (5.27) depend on microstructure length parameter λ. Action functional

$$A_{cb}\left(Q^k_\alpha, V^K\right) = \int\limits_0^{L_1}\int\limits_0^{L_2}\int\limits_{t_0}^{t_1} \langle L_{cb}\rangle \, dt d\xi dx, \tag{5.28}$$

where $\langle L_{cb}\rangle$ is given by (5.27), is called ***the tolerance averaging of action functional*** $A(u_{c\alpha}, w_b)$ ***defined by (5.26) under superimposed decomposition (5.22)***.

The principle of stationary action applied to A_{cb} given above leads to the following system of equations for Q^l_α, V^L

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial\langle L_{cb}\rangle}{\partial \dot{Q}^k_\alpha} - \frac{\partial\langle L_{cb}\rangle}{\partial Q^k_\alpha} + \partial_2 \frac{\partial\langle L_{cb}\rangle}{\partial\left(\partial_2 Q^k_\alpha\right)} = 0,\\
&\frac{\partial}{\partial t}\frac{\partial\langle L_{cb}\rangle}{\partial \dot{V}^K} - \frac{\partial\langle L_{cb}\rangle}{\partial V^K} + \partial_2 \frac{\partial\langle L_{cb}\rangle}{\partial\left(\partial_2 V^K\right)} - \partial_{22}\frac{\partial\langle L_{cb}\rangle}{\partial\left(\partial_{22} V^K\right)} = 0.
\end{aligned} \tag{5.29}$$

Combining (5.29) with (5.27) we obtain finally the explicit form of the Euler-Lagrange equations

$$\begin{aligned}
&\left\langle D^{\alpha 22\delta} c^k c^l\right\rangle \partial_{22} Q^l_\delta - \left\langle D^{\alpha 11\delta}\partial_1 c^k \partial_1 c^l\right\rangle Q^l_\delta - \underline{\left\langle \mu c^k c^l\right\rangle} a^{\alpha\beta}\ddot{Q}^l_\beta = \\
&\quad = r^{-1}\left\langle D^{\alpha 111}\partial_1 c^k w_0\right\rangle + \left\langle D^{\alpha\beta\gamma 1}\partial_1 c^k \partial_\beta u_{0\gamma}\right\rangle, \quad k,l = 1,2,\ldots,m,
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
&\underline{\left\langle B^{2222} b^K b^L\right\rangle}\partial_{2222} V^L + \left[\underline{\left\langle B^{1122} b^K \partial_{11} b^L\right\rangle} + \underline{\left\langle B^{1122} b^L \partial_{11} b^K\right\rangle} + \right.\\
&\quad \left. - 4\underline{\left\langle B^{1212}\partial_1 b^K \partial_1 b^L\right\rangle}\right]\partial_{22} V^L + \left\langle B^{1111}\partial_{11} b^K \partial_{11} b^L\right\rangle V^L + \underline{\left\langle \mu b^K b^L\right\rangle}\ddot{V}^L = \\
&\quad = -\left\langle B^{\alpha\beta 11}\partial_{11} b^K \partial_{\alpha\beta} w_0\right\rangle, \quad K,L = 1,2,\ldots,M.
\end{aligned} \tag{5.31}$$

Equations (5.30) and (5.31) together with *the micro-macro decomposition* (5.22) constitute ***the superimposed microscopic model*** (i.e. microscopic model imposed on the macroscopic model obtained in the first step of combined modelling). Coefficients of the derived model equations are *continuous and slowly-varying* in x and some of them *depend on a cell size* λ (underlined terms). The right-hand sides of (5.30) and (5.31) are known under assumption that $u_{0\alpha}$, w_0 were determined in the first step of modelling. The basic unknowns Q^k_α, V^K of the model equations must be *the slowly-varying functions in the tolerant periodicity direction*. The boundary conditions for Q^k_α, V^K should be defined only on boundaries $\xi = 0$, $\xi = L_2$. Let us observe that in the problem under consideration we have obtained system of governing equations which consists of *two independent subsystems*. The first from them is the system of $2m$ equations for fluctuation amplitudes Q^k_α, cf. (5.30), whereas the second one is the system of M equations for fluctuation amplitudes V^K, cf. (5.31).

Equations (5.30), (5.31) have to be considered together with decomposition

$$\begin{aligned}
&u_\alpha(x,\xi,t) = u^0_\alpha(x,\xi,t) + h^a(x)U^a_\alpha(x,\xi,t) + c^k(x)Q^k_\alpha(x,\xi,t),\\
&w(x,\xi,t) = w^0(x,\xi,t) + g^A(x)W^A(x,\xi,t) + b^K(x)V^K(x,\xi,t),\\
&\quad x \in \Omega, \quad (\xi,t) \in \Xi \times \mathrm{I}, \quad a = 1,2,\ldots,n, \quad k = 1,2,\ldots,m,\\
&\quad A = 1,2,\ldots,N, \quad K = 1,2,\ldots,M,
\end{aligned} \tag{5.32}$$

where functions u^0_α, U^a_α, w^0, W^A have to be obtained in the first step of combined modelling, i.e. in the framework of *the consistent asymptotic modelling*.

Combined asymptotic-tolerance model equations

Summarizing results obtained in ***Step 1*** and ***Step 2*** we conclude that *the combined asymptotic-tolerance model of selected dynamic problems for the tolerance-periodic shells under consideration* derived here is represented by

- *macroscopic model* defined by equations (5.20) for macrodisplacements u^0_α, w^0 with expressions (5.16) for fluctuation amplitudes U^a_α, W^A, $a = 1,2,\ldots,n$, $A = 1,2,\ldots,N$, obtained by means of ***the consistent asymptotic modelling*** and being independent of the microstructure length; it is assumed that in the framework of this model the solutions (5.22) to the problem under consideration are known,
- *superimposed microscopic model equations* (5.30), (5.31) derived by means of ***the tolerance (non-asymptotic) modelling***, some coefficients of these equations (underlined terms) depend on the microstructure length parameter λ,
- decomposition (5.32).

Coefficients of all equations derived in the framework of combined modelling *are continuous and slowly-varying* in $x \in \Omega$ in contrast to coefficients in starting equations (4.10), which are discontinuous, highly oscillating and tolerance-periodic. Moreover, some of them *depend on a cell size* λ (underlined terms).

Superimposed microscopic model equations independent of solutions obtained in the framework of macroscopic (asymptotic) model

Now, let us discuss an important modification of equations (5.30), (5.31). Let us assume $n \equiv m$, $N \equiv M$ and replace fluctuation shape functions $c^k(\cdot)$, $b^K(\cdot)$

in (5.30), (5.31) by fluctuation shape functions $h^a(\cdot)$, $a = 1,2,\ldots,n$, $g^A(\cdot)$, $A = 1,2,\ldots,N$, respectively. By means of the consistent asymptotic modelling we obtain

$$
\begin{aligned}
&\left\langle D^{\alpha 1\beta\gamma}\partial_1 h^a \partial_\beta u_{0\gamma}\right\rangle + r^{-1}\left\langle D^{\alpha 111}\partial_1 h^a w_0\right\rangle = \left\langle D^{\alpha 1\beta\gamma}\partial_1 h^a\right\rangle \partial_\beta u^0_\gamma + \\
&+ \left\langle D^{\alpha 11\gamma}\partial_1 h^a \partial_1 h^b\right\rangle U^b_\gamma + r^{-1}\left\langle D^{\alpha 111}\partial_1 h^a\right\rangle w^0 = 0, \\
&\left\langle B^{\alpha\beta 11}\partial_{11} g^A \partial_{\alpha\beta} w_0\right\rangle = \left\langle B^{\alpha\beta 11}\partial_{11} g^A\right\rangle \partial_{\alpha\beta} w^0 + \\
&+ \left\langle B^{1111}\partial_{11} g^A \partial_{11} g^B\right\rangle W^B = 0, \\
&a,b = 1,2,\ldots,n, \quad A,B = 1,2,\ldots,N.
\end{aligned}
\tag{5.33}
$$

From comparison of $(5.33)_1$ with (5.30) and $(5.33)_2$ with (5.31) it follows that the right-hand sides of equations (5.30), (5.31) are equal to zero. Moreover, taking into account a symmetric form of tensor $D^{\alpha\beta\gamma\delta}$ we arrive finally to the following equations for unknown fluctuation amplitudes $Q^b_1(x,\xi,t)$, $Q^b_2(x,\xi,t)$, $V^B(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$,

$$
\begin{gathered}
\underline{\left\langle D^{1221} h^a h^b\right\rangle}\partial_{22} Q^b_1 - \left\langle D^{1111}\partial_1 h^a \partial_1 h^b\right\rangle Q^b_1 - \underline{\left\langle \mu h^a h^b\right\rangle}\ddot{Q}^b_1 = 0, \\
a,b = 1,2,\ldots,n,
\end{gathered}
\tag{5.34}
$$

$$
\begin{gathered}
\underline{\left\langle D^{2222} h^a h^b\right\rangle}\partial_{22} Q^b_2 - \left\langle D^{2112}\partial_1 h^a \partial_1 h^b\right\rangle Q^b_2 - \underline{\left\langle \mu h^a h^b\right\rangle}\ddot{Q}^b_2 = 0, \\
a,b = 1,2,\ldots,n,
\end{gathered}
\tag{5.35}
$$

$$
\begin{aligned}
&\underline{\left\langle B^{2222} g^A g^B\right\rangle}\partial_{2222} V^B + \Big(\underline{\left\langle B^{1122} g^A \partial_{11} g^B\right\rangle} + \underline{\left\langle B^{1122} g^B \partial_{11} g^A\right\rangle} + \\
&-4\underline{\left\langle B^{1212}\partial_1 g^A \partial_1 g^B\right\rangle}\Big)\partial_{22} V^B + \left\langle B^{1111}\partial_{11} g^A \partial_{11} g^B\right\rangle V^B + \\
&+ \underline{\left\langle \mu g^A g^B\right\rangle}\ddot{V}^B = 0, \quad A,B = 1,2,\ldots,N.
\end{aligned}
\tag{5.36}
$$

Equations (5.34)-(5.36) are independent of solutions $u_{0\alpha}$, w_0, obtained in the framework of *the macroscopic model* and hence ***describe selected problems of the shell micro-dynamics*** (e.g. the free micro-vibrations, propagation of waves related to the micro-fluctuation amplitudes) ***independently of the shell macro-dynamics***. Moreover, in the problem considered here, the micro-dynamic behaviour of the shells in the axial, circumferential and normal directions can be analysed independently of each other.

Discussion of results

The characteristic features of the proposed ***combined asymptotic-tolerance model for the analysis of selected dynamic problems for thin cylindrical shells with a tolerance-periodic microstructure and a transversally graded macrostructure in the circumferential direction*** are:

- The combined model equations consist of *macroscopic model equations* (5.20) formulated by means of *the consistent asymptotic procedure* which are combined with *superimposed microscopic model equations* (5.30), (5.31) derived by applying *the tolerance modelling technique* and under assumption that in the framework of the macroscopic model the solutions (5.22) to the problem under consideration are known.

- In contrast to starting equations (4.10) with discontinuous, highly oscillating and tolerance-periodic coefficients, the combined model equations proposed here ***have continuous and slowly-varying coefficients***. Moreover, ***some coefficients of the superimposed microscopic model equations depend on a cell size*** λ. Thus, the combined model can be applied to the analysis of many phenomena caused by the length-scale effect.

- The resulting combined model equations are uniquely determined by the highly oscillating tolerance-periodic ***fluctuation shape functions***, which have to be known in every problem under consideration. In general case, the fluctuation shape functions of both the macroscopic and the microscopic models are different. Under assumption that the fluctuation shape functions of both the models coincide, we have derived superimposed microscopic model equations (5.34)-(5.36) which are independent of the solutions obtained in the framework of the macroscopic model. Taking into account this result we can conclude that ***an important advantage of the combined model is that it makes it possible to separate the macroscopic description of some special problems from their microscopic description***. It means that in the framework of the combined model we can study micro-dynamics of periodic shells under consideration independently of their macro-dynamics.

- It can be shown that equations (5.34)-(5.36) also describe certain ***near-initial and near-boundary phenomena*** strictly related to the specific form of initial conditions and boundary conditions on $\Omega \times \partial\Xi$, where $\Xi = (0, L_2)$. That is why, equations (5.34)-(5.36) are referred to as *the boundary layer equations*, where the term "boundary" is related both to time and space.

- Applying *the tolerance modelling* directly to the decomposition (5.32) we also obtain the system of equations for u_α^0, w^0, U_α^a, Q_α^k, W^A, V^K. However, this system is much more complicated than the system obtained in the framework of the combined modelling.

6. Selected problems of dynamics: Application of the tolerance and asymptotic models

6.1. Introduction

In all dynamic problems investigated in Section 6 and also in the subsequent Section 7, the object of considerations is a thin cylindrical shell with L_1, L_2, r, d as its circumferential length, axial length, midsurface curvature radius and constant thickness, respectively. The shell has a functionally graded macrostructure and a tolerance-periodic microstructure along circumferential direction as well as a constant structure in the axial direction. On the microscopic level, the shell is made of two elastic isotropic materials, which are perfectly bonded on interfaces and tolerance-periodically distributed along x-coordinate. Such a shell is shown in Fig. 4.1 and reminded in Fig. 6.1.

The basic cell Δ shown in Fig. 6.2 and cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ are defined by (4.1). Below we recall the definitions

$$
\begin{aligned}
&\Delta \equiv \left[-\lambda/2, \lambda/2\right], \\
&\Delta(x) \equiv x + \Delta = \left[x - \lambda/2, x + \lambda/2\right], \\
&x \in \Omega_\Delta, \quad \Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega\},
\end{aligned}
\tag{6.1}
$$

where λ is a cell length dimension in $x \equiv x^1$-direction, cf. Figs. 6.1 and 6.2. *The microstructure length parameter* λ satisfies conditions: $\lambda/\max(d) \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$. Setting $z \in [-\lambda/2, \lambda/2]$, we assume that the cell Δ has a symmetry axis for $z = 0$. Inside the cell the geometrical, elastic and inertial properties of the shell are described by symmetric (i.e. even) functions of argument z. At the same time, these functions are independent of argument $\xi \equiv x^2$.

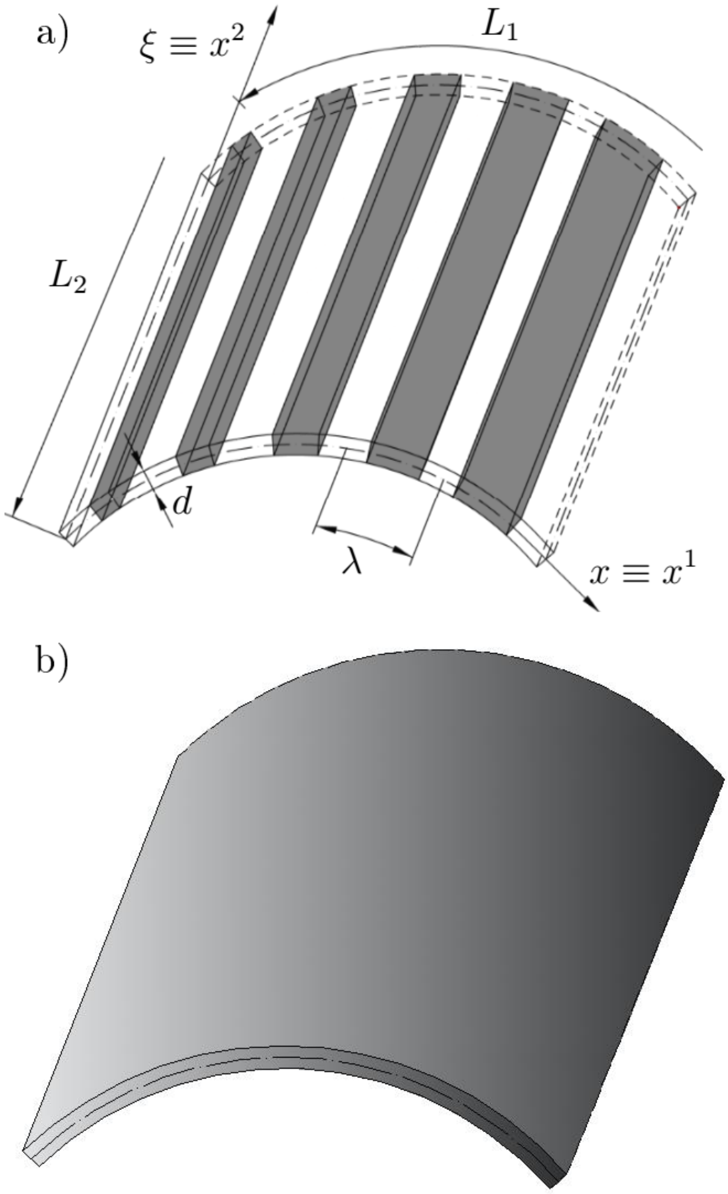

Figure 6.1: Fragment of the shell made of tolerance-periodically distributed two component materials a) the microscopic point of view b) the macroscopic point of view

Properties of the component materials are described by: Young's moduli E_1, E_2, Poisson's ratios ν_1, ν_2 and mass densities ρ_1, ρ_2, cf. Fig. 6.2. It is assumed that elastic $E(x)$ and inertial $\rho(x)$ properties of the composite shell are tolerance-periodic functions in x, $x \in \Omega$, i.e. $E, \rho \in TP^0_\delta(\Omega, \Delta)$, but Poisson's ratio $\nu \equiv \nu_1 = \nu_2$ is constant. Inside the cell, periodic approximations $\widetilde{E}(x, z)$, $\widetilde{\rho}(x, z)$, $x \in \Delta(x)$, $x \in \Omega_\Delta$ of functions $E(\cdot)$, $\rho(\cdot)$ take the form

$$\widetilde{E}(\cdot,z),\widetilde{\rho}(\cdot,z)=\begin{cases} E_1,\rho_1 & \text{for } z\in\left(-\widetilde{\eta}(x)\lambda/2,\widetilde{\eta}(x)\lambda/2\right), \\ E_2,\rho_2 & \text{for } z\in\left[-\lambda/2,-\widetilde{\eta}(x)\lambda/2\right]\cup\left[\widetilde{\eta}(x)\lambda/2,\lambda/2\right], \end{cases} \tag{6.2}$$

where $\widetilde{\eta}(x)\in[0,1]$ is a periodic approximation of function $\eta(x)\in[0,1]$ describing distribution of material properties, cf. Fig. 6.2.

The rigidities $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x\in\Omega$, $D^{\alpha\beta\gamma\delta}, B^{\alpha\beta\gamma\delta}\in TP^0_\delta(\Omega,\Delta)$, of the shell are described by: $D^{\alpha\beta\gamma\delta}(x)=D(x)H^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}(x)=B(x)H^{\alpha\beta\gamma\delta}$, where $D(x)=E(x)d/(1-\nu^2)$, $B(x)=E(x)d^3/(12(1-\nu^2))$ and the nonzero components of tensor $H^{\alpha\beta\gamma\delta}$ are: $H^{1111}=H^{2222}=1$, $H^{1122}=H^{2211}=\nu$, $H^{1212}=H^{1221}=H^{2121}=H^{2112}=(1-\nu)/2$. Periodic approximations of these rigidities in the cell take the form: $\widetilde{D}^{\alpha\beta\gamma\delta}(x,z)=\widetilde{E}(x,z)\ d\ (1-\nu^2)^{-1}H^{\alpha\beta\gamma\delta}$, $\widetilde{B}^{\alpha\beta\gamma\delta}(x,z)=\widetilde{E}(x,z)\ d^3\ (12(1-\nu^2))^{-1}H^{\alpha\beta\gamma\delta}$, $z\in\Delta(x)$, $x\in\Omega_\Delta$, where $\widetilde{E}(x,z)$ is given by (6.2).

The shell mass density $\mu(x)$ per midsurface unit area and its periodic approximation $\widetilde{\mu}(x,z)$ in the cell are given by $\mu(x)=\rho(x)d$ and $\widetilde{\mu}(x,z)=\widetilde{\rho}(x,z)d$, respectively, where $\widetilde{\rho}(x,z)$ is given by (6.2).

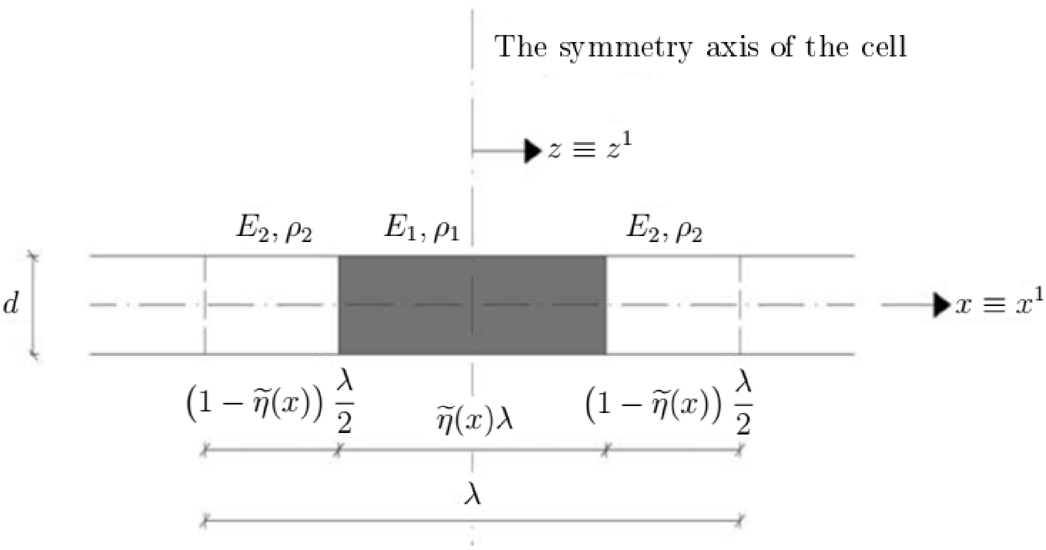

Figure 6.2: Basic cell $\Delta\equiv[-\lambda/2,\lambda/2]$ of the tolerance-periodic shell

The considerations will be restricted to the simplest forms of the tolerance, asymptotic and asymptotic-tolerance models in which $a=n=A=N=1$. It means that we will take into account only one fluctuation shape function $h(x)\equiv h^1(x)$, $x\in\Omega$, $h\in FS^1_\delta(\Omega,\Delta)$, which periodic approximation $\widetilde{h}(x,z)\equiv\widetilde{h}^1(x,z)$, $z\in\Delta(x)$, $x\in\Omega_\Delta$ is antisymmetric on the cell (i.e. odd with respect to argument z), and only one fluctuation shape function $g(x)\equiv g^1(x)$, $x\in\Omega$, $g\in FS^2_\delta(\Omega,\Delta)$, which periodic approximation $\widetilde{g}(x,z)\equiv\widetilde{g}^1(x,z)$, $z\in\Delta(x)$,

$x \in \Omega_\Delta$, is symmetric on the cell (i.e. even with respect to argument z). The fluctuation shape functions should approximate the expected principal modes of the shell free vibrations on a cell. On a basis of knowledge of these principal modes in thin heterogeneous shells and plates, cf. e.g. Tomczyk [133], Jędrysiak [27], we shall postulate fluctuation shape functions in the form of trigonometric functions. We shall introduce the following periodic approximations $\widetilde{h}(x,z)$, $\widetilde{g}(x,z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, of fluctuation shape functions $h(x)$, $g(x)$, $x \in \Omega$,

$$\widetilde{h}(x,z) = \lambda \sin\left(2\pi z/\lambda\right), \tag{6.3}$$

$$\widetilde{g}(x,z) = \lambda^2 \left[\cos\left(2\pi z/\lambda\right) + c(x)\right], \tag{6.4}$$

where $c(x)$ is a slowly-varying function in x and is determined by condition $\langle\widetilde{\mu}\widetilde{g}\rangle = 0$

$$c(x) = -\frac{(\rho_1 - \rho_2)\sin\left(\pi(\widetilde{\eta}(x)\right)}{\pi\left(\rho_1\widetilde{\eta}(x) + \rho_2(1-\widetilde{\eta}(x)\right)} \tag{6.5}$$

with $\widetilde{\eta}(x) \in [0,1]$ being a periodic approximation of function $\eta(x)$ describing distribution of material properties.

Function $c(x)$ is treated as constant in calculations of derivatives $\partial_1\widetilde{g}$, $\partial_{11}\widetilde{g}$.

In the calculational examples considered in the application part of this dissertation, i.e. in Sections 6 and 7, the following periodic approximations $\widetilde{\eta}(x)$ of distribution functions of material properties $\eta(x)$ will be taken into account

$$\widetilde{\eta}(x) = x/L, \tag{6.6}$$

$$\widetilde{\eta}(x) = \left(x/L\right)^2, \tag{6.7}$$

$$\widetilde{\eta}(x) = \left(2x/L - 1\right)^2, \tag{6.8}$$

$$\widetilde{\eta}(x) = \left(x/L\right)^3, \tag{6.9}$$

$$\widetilde{\eta}(x) = \sin\left(\pi x/L\right), \tag{6.10}$$

$$\widetilde{\eta}(x) = \cos\left(\pi x/(2L)\right), \tag{6.11}$$

$$\widetilde{\eta}(x) = \sin^2\left(\pi x/L\right), \tag{6.12}$$

$$\widetilde{\eta}(x) = \cos^2\left(\pi x/L\right), \tag{6.13}$$

$$\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2, \tag{6.14}$$

$$\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L), \tag{6.15}$$

$$\widetilde{\eta}(x) = \eta = 0.5, \tag{6.16}$$

where $L \equiv L_1$ and where function $\widetilde{\eta}(x) = 0.5$ describes periodic distribution of material properties along circumferential direction.

Diagrams of functions (6.6)-(6.16) are shown in Fig. 6.3.

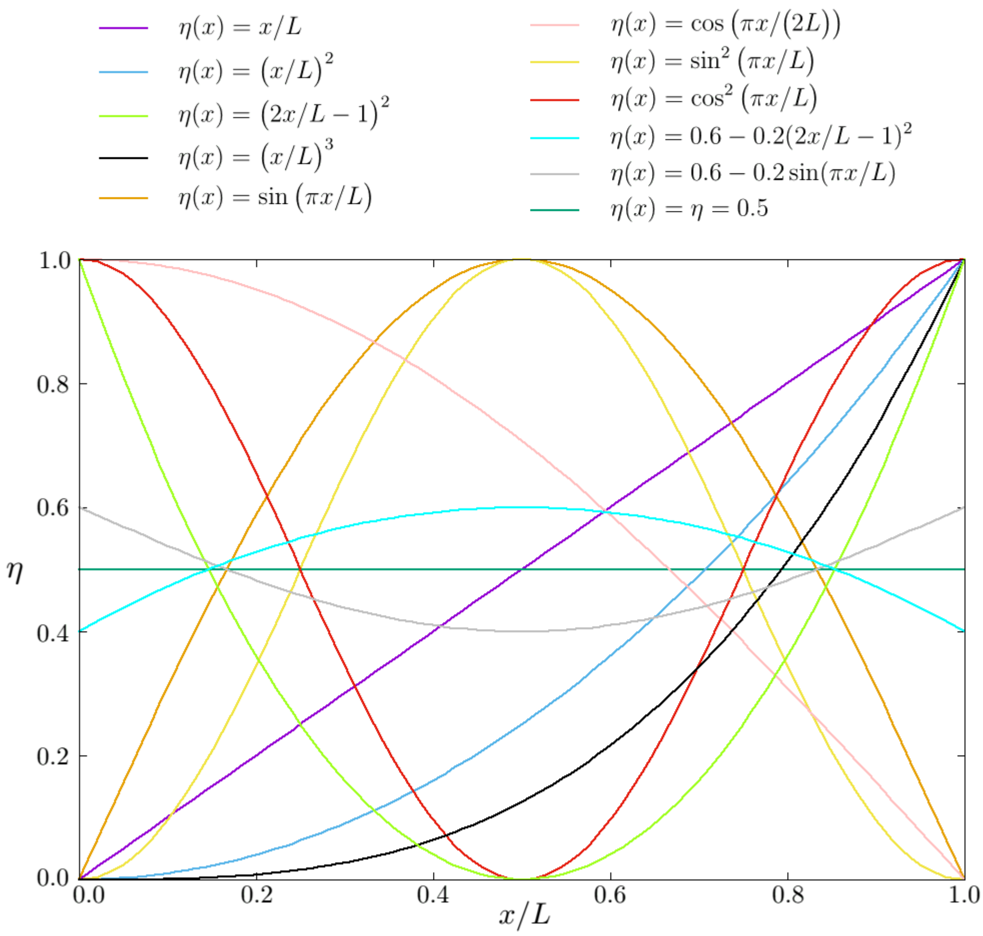

Figure 6.3: Diagrams of functions $\eta(x) \in [0, 1]$ describing distribution of material properties, which periodic approximations are expressed by (6.6)-(6.16)

In the sequel, under assumption that for the shells under consideration condition $\lambda/r \ll 1$ holds we shall introduce the extra approximation $1 + \lambda/r \approx 1$.

6.2. Free vibrations of the transversally graded shell strip

6.2.1 Formulation of the problem

In this subsection transversal free vibrations of a thin simply supported shell strip with span $L \equiv L_1$ along the circumferential $x \equiv x^1$-coordinate and with constant thickness are discussed. The shell strip has a tolerance-periodic microstructure and a functionally graded macrostructure along its span as well as constant structure in the axial direction. It assumed that the shell strip is made of two elastic isotropic materials, which are perfectly bonded on interfaces and densely, tolerance-periodically distributed along x-coordinate. A fragment of such a shell strip is shown in Fig. 6.1, where in the problem under consideration length dimension L_2 of the shell along $\xi \equiv x^2$-coordinate is assumed to be infinite.

The basic cell defined by $\Delta \equiv [-\lambda/2, \lambda/2]$, cf. definition (6.1), is shown in Fig. 6.2.

Properties of the component materials are described by Young's moduli E_1, E_2, Poisson's ratio $\nu \equiv \nu_1 = \nu_2$ and mass densities ρ_1, ρ_2. Inside the cell, the elastic $E \in TP^0_\delta(\Omega, \Delta)$ and inertial $\rho \in TP^0_\delta(\Omega, \Delta)$ properties of the shell strip have periodic approximations $\widetilde{E}(x, z)$, $\widetilde{\rho}(x, z)$ $z \in \Delta(x)$, $x \in \Omega_\Delta$, defined by (6.2).

The rigidities $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$ of the shell strip are described in Subsection 6.1.

The considerations will be based on equations (5.6), (5.7) of the tolerance model and equations (5.18) of the asymptotic model and restricted to the simplest forms of these models in which $a = n = A = N = 1$.

In order to investigate free vibrations, we assume that external forces f^a, f are equal to zero.

Moreover, the forces of inertia in directions tangential to the shell midsurface are neglected.

We also neglect fluctuating parts hU_α of displacements u_α.

Periodic approximation $\widetilde{g}(x, z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, of fluctuation shape function $g \in FS^2_\delta(\Omega, \Delta)$ is given by (6.4).

This dynamic problem is treated to be independent of the ξ-coordinate. Hence, $u^0_2 = 0$ and the remaining unknowns u^0_1, w^0, W of the tolerance and asymptotic models proposed in this dissertation are only functions of x-midsurface parameter and t-coordinate.

The investigations will be carried out for different material properties distribution functions.

Bearing in mind assumptions given above, the effect of a cell size on free vibration frequencies of the shell strip under consideration will be analysed by using both the tolerance model represented by equations of motion (5.7) with constitutive relations (5.6) and the asymptotic model governed by equations (5.18).

Moreover, the influence of differences between elastic and inertial properties of the constituent materials on these frequencies will be studied.

6.2.2 Analysis in the framework of tolerance model

Now, the system of tolerance equations (5.7) reduces to the following system of three equations for $u_1^0(x,z)$, $w^0(x,t)$, $W(x,t)$, $(x,t) \in \Omega \times \mathrm{I}$

$$\begin{aligned}
&\partial_1 \left(\left\langle D^{1111} \right\rangle \partial_1 u_1^0 + r^{-1} \left\langle D^{1111} \right\rangle w^0 \right) = 0, \\
&\partial_{11} \left(\left\langle B^{1111} \right\rangle \partial_{11} w^0 + \left\langle B^{1111} \partial_{11} g \right\rangle W \right) + r^{-1} \left\langle D^{1111} \right\rangle \partial_1 u_1^0 + \\
&\quad + r^{-2} \left\langle D^{1111} \right\rangle w^0 + \langle \mu \rangle \ddot{w}^0 = 0, \\
&\left\langle B^{1111} \partial_{11} g \right\rangle \partial_{11} w^0 + \left\langle B^{1111} \left(\partial_{11} g\right)^2 \right\rangle W + \lambda^4 \left\langle \mu \left(\overline{g}\right)^2 \right\rangle \ddot{W} = 0,
\end{aligned} \tag{6.17}$$

where $\overline{g}(\cdot) = \lambda^{-2} g(\cdot)$. We recall that derivative $\partial_{11} g(x)$ of fluctuation shape function $g(x)$ is independent of λ as parameter. In equations (6.17) only the micro-inertia forces $\lambda^4 \left\langle \mu \left(\overline{g}\right)^2 \right\rangle \ddot{W}$ depend on microstructure length parameter λ. All coefficients of (6.17) are continuous and slowly-varying functions in argument x.

It is difficult to find analytical solutions to Eqs. (6.17). Thus, to obtain approximate formulas of free vibration frequencies the known Ritz method can be applied, cf. Kaliski [46]. Using this method, formulas of the maximal strain energy $E_{\max}$ and the maximal kinetic energy $K_{\max}$ are determined.

In the problem under consideration, strain energy function $E(x,t)$, $(x,t) \in \Omega \times \mathrm{I}$, related to the shell midsurface has the form

$$E = \frac{1}{2} \left(D^{1111} \left(\varepsilon_{11}\right)^2 + B^{1111} \left(\kappa_{11}\right)^2 \right), \tag{6.18}$$

where $\varepsilon_{11} = \partial_1 u_1 + r^{-1} w$ and $\kappa_{11} = -\partial_{11} w$.

The kinetic energy function $K(x,t)$, $(x,t) \in \Omega \times \mathrm{I}$, related to the shell midsurface is given by

$$K = \frac{1}{2} \mu \left(\dot{w}\right)^2. \tag{6.19}$$

Tolerance modelling applied to (6.18) and (6.19) yields the results

$$\begin{aligned}
\langle E \rangle (x) = \frac{1}{2} \Big[&\left\langle D^{1111} \right\rangle (x) \left(\left(\partial_1 u_1^0\right)^2 + r^{-2} \left(w^0\right)^2 + 2 r^{-1} \partial_1 u_1^0 w^0 \right) + \\
&+ \left\langle B^{1111} \right\rangle (x) \left(\partial_{11} w^0\right)^2 + 2 \left\langle B^{1111} \partial_{11} g \right\rangle (x) \partial_{11} w^0 W + \\
&+ \left\langle B^{1111} \left(\partial_{11} g\right)^2 \right\rangle (x) \left(W\right)^2 \Big], \quad x \in \Omega_\Delta,
\end{aligned} \tag{6.20}$$

and

$$\langle K\rangle(x)=\frac{1}{2}\left[\langle\mu\rangle(x)\left(\dot{w}^0\right)^2+\lambda^4\left\langle\mu\left(\overline{g}\right)^2\right\rangle(x)\left(\dot{W}\right)^2\right],\quad x\in\Omega_\Delta. \tag{6.21}$$

The maximal strain energy and the maximal kinetic energy will be obtained by integrating results (6.20) and (6.21) over region $\Omega=(0,L)$. Unknown functions in (6.20), (6.21) must satisfy the given boundary conditions for $x=0$, $x=L$.

For the shell strip the solutions to Eqs. (6.17) can be assumed in the form

$$\begin{aligned}u_1^0(x,t)&=A_1\Psi(\alpha x)\cos(\omega t),\\ w^0(x,t)&=A_2\Phi(\alpha x)\cos(\omega t),\\ W(x,t)&=A_3\Theta(\alpha x)\cos(\omega t),\end{aligned} \tag{6.22}$$

where α is a wave number, ω is a free vibration frequency of transverse free vibrations. Functions $\Psi(\cdot)$, $\Phi(\cdot)$, $\Theta(\cdot)$ have to satisfy the given boundary conditions for $x=0$, $x=L$. They relate to the principal free vibration modes.

Denote the first derivative of $\Psi(\cdot)$ and the second derivative of $\Phi(\cdot)$ by

$$\begin{aligned}\partial_1\Psi(\alpha x)&\equiv\alpha\overline{\Psi}(\alpha x),\\ \partial_{11}\Phi(\alpha x)&\equiv\alpha^2\overline{\Phi}(\alpha x).\end{aligned} \tag{6.23}$$

Moreover, let us introduce the following denotations

$$\begin{aligned}
\widehat{D}&\equiv\int_\Omega\left\langle D^{1111}\right\rangle(x)\left[\overline{\Psi}(\alpha x)\right]^2dx,\\
\widehat{D}^*&\equiv r^{-2}\int_\Omega\left\langle D^{1111}\right\rangle(x)\left[\Phi(\alpha x)\right]^2dx,\\
\widehat{D}^{**}&\equiv r^{-1}\int_\Omega\left\langle D^{1111}\right\rangle(x)\overline{\Psi}(\alpha x)\Phi(\alpha x)dx,\\
\widehat{B}&\equiv\int_\Omega\left\langle B^{1111}\right\rangle(x)\left[\overline{\Phi}(\alpha x)\right]^2dx,\\
\overline{B}&\equiv\int_\Omega\left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle(x)\left[\Theta(\alpha x)\right]^2dx,\\
\overline{B}^*&\equiv\int_\Omega\left\langle B^{1111}\partial_{11}g\right\rangle(x)\overline{\Phi}(\alpha x)\Theta(\alpha x)dx,\\
\widehat{\mu}&\equiv\int_\Omega\langle\mu\rangle(x)\left[\Phi(\alpha x)\right]^2dx,\\
\overline{\mu}&\equiv\int_\Omega\left\langle\mu\left(\overline{g}\right)^2\right\rangle(x)\left[\Theta(\alpha x)\right]^2dx,
\end{aligned} \tag{6.24}$$

where averages $\left\langle D^{1111}\right\rangle(x)$, $\left\langle B^{1111}\right\rangle(x)$, $\left\langle B^{1111}\partial_{11}g\right\rangle(x)$, $\left\langle B^{1111}(\partial_{11}g)^2\right\rangle(x)$, $\langle\mu\rangle(x)$, $\left\langle\mu(\overline{g}^2\right\rangle(x)$ are given in Appendix, cf. (A.1), (A.8), (A.11), (A.13), (A.19) and (A.21). We recall that $\overline{g}=\lambda^{-2}g$.

Taking into account (6.22) and using denotations (6.24), the maximal strain energy $E_{\max}$ and the maximal kinetic energy $K_{\max}$ by the tolerance model can be written as

$$
\begin{aligned}
E_{\max} = \frac{1}{2}\Big[& \alpha^4\widehat{B}\,(A_2)^2 + \alpha^2\left(2\overline{B}^*A_2A_3 + \widehat{D}\,(A_1)^2\right) + \\
& + 2\alpha\widehat{D}^{**}A_1A_2 + \overline{B}\,(A_3)^2 + \widehat{D}^*\,(A_2)^2\Big], \\
K_{\max} = \frac{1}{2} & \left[\widehat{\mu}\,(A_2)^2 + \lambda^4\overline{\mu}\,(A_3)^2\right]\omega^2.
\end{aligned}
\tag{6.25}
$$

Substituting the right-hand sides of (6.25) into the conditions of the Ritz method

$$
\begin{aligned}
\frac{\partial\left(E_{\max}-K_{\max}\right)}{\partial A_1} &= 0, \\
\frac{\partial\left(E_{\max}-K_{\max}\right)}{\partial A_2} &= 0, \\
\frac{\partial\left(E_{\max}-K_{\max}\right)}{\partial A_3} &= 0,
\end{aligned}
\tag{6.26}
$$

we obtain from (6.26) the system of three linear homogeneous algebraic equations for A_i, $i=1,2,3$. For a non-trivial solution, the determinant of this system must be equal to zero. In this manner we arrive at the characteristic equation for frequency ω of the transverse free vibrations of the shell strip under consideration. Under extra denotations

$$
\begin{aligned}
\overline{d} &\equiv -\left(\widehat{D}^{**}\right)^2\left(\widehat{D}\right)^{-1} + \widehat{D}^*, \\
\overline{\varepsilon} &\equiv \left(\frac{\lambda}{L}\right)^4,
\end{aligned}
\tag{6.27}
$$

from the characteristic equation mentioned above we derive the following formulae for ***the fundamental lower free vibration frequency*** ω_- and for ***the new additional higher free vibration frequency*** ω_+, caused by a tolerance-periodic structure of the shell strip

$$(\omega_-)^2 = \frac{1}{2}\left(\frac{\overline{B}}{L^4\overline{\mu}\ \overline{\varepsilon}} + \frac{\alpha^4\widehat{B} + \overline{d}}{\widehat{\mu}}\right) +$$

$$- \frac{\overline{B}}{2L^4\overline{\mu}\ \overline{\varepsilon}}\left[1 + \left(\frac{4\pi^4\left(\overline{B}^*\right)^2}{\widehat{\mu}\overline{B}^2} - 2\frac{\pi^4\widehat{B} + \overline{d}L^4}{\widehat{\mu}\overline{B}}\right)\overline{\mu}\ \overline{\varepsilon} + \right. \tag{6.28}$$

$$\left. + \left(\frac{\pi^8\widehat{B}^2 + 2\pi^4L^4\widehat{B}\overline{d} + L^8\overline{d}^2}{\widehat{\mu}\overline{B}}\right)\overline{\mu}^2\overline{\varepsilon}^2\right]^{1/2},$$

$$(\omega_+)^2 = \frac{1}{2}\left(\frac{\overline{B}}{L^4\overline{\mu}\ \overline{\varepsilon}} + \frac{\alpha^4\widehat{B} + \overline{d}}{\widehat{\mu}}\right) +$$

$$+ \frac{\overline{B}}{2L^4\overline{\mu}\ \overline{\varepsilon}}\left[1 + \left(\frac{4\pi^4\left(\overline{B}^*\right)^2}{\widehat{\mu}\overline{B}^2} - 2\frac{\pi^4\widehat{B} + \overline{d}L^4}{\widehat{\mu}\overline{B}}\right)\overline{\mu}\ \overline{\varepsilon} + \right. \tag{6.29}$$

$$\left. + \left(\frac{\pi^8\widehat{B}^2 + 2\pi^4L^4\widehat{B}\overline{d} + L^8\overline{d}^2}{\widehat{\mu}\overline{B}}\right)\overline{\mu}^2\overline{\varepsilon}^2\right]^{1/2},$$

where constant $\overline{\varepsilon}$ involves a microstructure length parameter λ.

It can be observed that $\overline{\varepsilon} \equiv (\lambda/L)^4 \ll 1$ can be treated as a small parameter. Representing the square root in (6.28) in the form of Maclaurin power series with respect to $\overline{\varepsilon}$, we obtain the following approximate formula for ω_-^2

$$(\omega_-)^2 = \frac{1}{\widehat{\mu}}\left[\alpha^4\left(\widehat{B} - \frac{\left(\overline{B}^*\right)^2}{\overline{B}}\right) + \overline{d}\right] + O\left(\overline{\varepsilon}\right), \tag{6.30}$$

where $O\left(\overline{\varepsilon}\right) \to 0$ together with $\overline{\varepsilon} \to 0$. From result (6.30) it follows that *the lower free vibration frequency* ω_- is independent of a cell size λ, contrary to *the higher free vibration frequency* ω_+ which depends on λ.

6.2.3 Analysis in the framework of asymptotic model

In order to evaluate obtained results, let us consider the above problem within the asymptotic model. Now, asymptotic model equations (5.18) reduce to the following form

$$
\begin{aligned}
&\partial_1 \left(\left\langle D^{1111} \right\rangle \partial_1 u_1^0 + r^{-1} \left\langle D^{1111} \right\rangle w^0 \right) = 0, \\
&\partial_{11} \left(\left\langle B^{1111} \right\rangle \partial_{11} w^0 + \left\langle B^{1111} \partial_{11} g \right\rangle W \right) + r^{-1} \left\langle D^{1111} \right\rangle \partial_1 u_1^0 + \\
&\quad + r^{-2} \left\langle D^{1111} \right\rangle w^0 + \langle \mu \rangle \ddot{w}^0 = 0, \\
&\left\langle B^{1111} \partial_{11} g \right\rangle \partial_{11} w^0 + \left\langle B^{1111} \left(\partial_{11} g \right)^2 \right\rangle W = 0,
\end{aligned}
\tag{6.31}
$$

Note, that (6.31) can also be directly derived from governing equations (6.17) by neglecting micro-inertia forces $\lambda^4 \left\langle \mu (\overline{g})^2 \right\rangle \ddot{W}$ depending explicitly on microstructure length parameter λ.

Asymptotic modelling applied to strain energy (6.18) yields the result which coincides with result (6.20) obtained in the framework of the tolerance modelling.

Asymptotic modelling applied to kinetic energy (6.19) leads to the following formula for $\langle K \rangle (x)$

$$
\langle K \rangle (x) = \frac{1}{2} \langle \mu \rangle (x) \left(\dot{w}^0 \right)^2 . \tag{6.32}
$$

Assuming solutions to (6.31) in the form of (6.22) and using denotations (6.24), we derive formulas for the maximal strain energy and the maximal kinetic energy. The maximal strain energy obtained in the framework of asymptotic model coincides with corresponding result $(6.25)_1$ derived within the tolerance model. The maximal kinetic energy has the form

$$
K_{\max} = \frac{1}{2} \widehat{\mu} \left(A_2 \right)^2 \left(\omega^{AM} \right)^2 . \tag{6.33}
$$

Substituting the right-hand sides of $(6.25)_1$ and (6.33) into conditions (6.26) of the Ritz method, after some manipulations the formula for frequency ω^{AM} of the shell's transverse free vibrations is obtained in the framework of the asymptotic model under consideration

$$
\left(\omega^{AM} \right)^2 = \frac{1}{\widehat{\mu}} \left[\alpha^4 \left(\widehat{B} - \frac{\left(\overline{B}^* \right)^2}{\overline{B}} \right) + \overline{d} \right] , \tag{6.34}
$$

This frequency is independent of a cell size.

6.2.4 Discussion of analytical results

Analysing results obtained in *6.2.2* and *6.2.3* the following important conclusions can be formulated:

- *In the framework of the tolerance model, not only the fundamental lower ω_-, but also the new additional higher ω_+ free vibration frequencies can be derived and analysed*; cf. (6.28), (6.29). The higher free vibration frequency is caused by a tolerance-periodic microstructure of the shell strip under consideration and hence it depends on a microstructure length parameter λ. This frequency cannot be determined using the asymptotic model.
- Comparing (6.30) and (6.34), we arrive at the following interrelation between $(\omega_-)^2$ and $(\omega^{AM})^2$

$$(\omega_-)^2 = \left(\omega^{AM}\right)^2 + O\left(\lambda^4\right) \tag{6.35}$$

It means, that differences between the values of lower free vibration frequency ω_- derived from the tolerance model and free vibration frequency ω^{AM} obtained from the asymptotic one are negligibly small. Thus, in the problem under consideration, *the effect of microstructure length parameter λ on the "classical" free vibration frequencies can be neglected.* It means that *the asymptotic model governed by equations* (6.31) *is sufficient to determine and investigate free vibration frequencies of the micro-heterogeneous cylindrical shell strip under consideration.*

6.2.5 Numerical calculations

The shell strip simply supported on both edges is taken into account.

Functions $\Psi(\cdot)$, $\Phi(\cdot)$, $\Theta(\cdot)$ occurring in solutions (6.22) satisfying boundary conditions for the shell strip simply supported on edges $x = 0$, $x = L$, i.e. boundary conditions, cf. Kaliski [46],

$$\begin{aligned}
&\partial_1\Psi(0) = \Phi(0) = \Theta(0) = \partial_{11}\Phi(0) = \partial_{11}\Theta(0) = 0,\\
&\partial_1\Psi(L) = \Phi(L) = \Theta(L) = \partial_{11}\Phi(L) = \partial_{11}\Theta(L) = 0,
\end{aligned}$$

are assumed in the form

$$\begin{aligned}
&\Psi(\alpha x) = \cos\alpha x,\\
&\Phi(\alpha x) = \Theta(\alpha x) = \sin(\alpha x),
\end{aligned} \tag{6.36}$$

where the wave number α is equal to π/L.

Calculations are made for approximations $\widetilde{\eta}(x)$ of distribution functions of material properties $\eta(x)$ given by (6.6)-(6.13) and (6.16).

Diagrams of these functions are shown in Fig. 6.3.

We define the following dimensionless free vibration frequencies

$$(\Omega_-)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\omega_-\right)^2, \tag{6.37}$$

$$\left(\Omega_+\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\omega_+\right)^2, \tag{6.38}$$

$$\left(\Omega^{AM}\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\omega^{AM}\right)^2, \tag{6.39}$$

where frequencies ω_-, ω_+, ω^{AM} are determined by formulae (6.28), (6.29) and (6.34), respectively.

Some numerical results calculated by formulae (6.37)-(6.39) are shown in Figs. 6.4-6.22.

Calculations are made for Poisson ratio $\nu = 0.3$, for fixed ratio $d/\lambda = 0.1$ and for various ratios $\varepsilon \equiv \lambda/L \in [0.01, 0.1]$, $E_2/E_1 \in [0.2, 1.0]$, $\rho_2/\rho_1 \in [0.2, 1.0]$.

All plots are made under assumption $L = const$. It means that the variations of $\varepsilon \equiv \lambda/L$ are caused by the changes of a cell size λ. Moreover, for fixed geometrical ratio $d/\lambda = 0.1$, the variations of λ imply the changes of shell thickness d with respect to L, i.e. $d/\lambda = d/(\varepsilon L) = 0.1$ and hence $d/L = 0.1\varepsilon$.

In Figs. 6.4, 6.5 there are presented diagrams of dimensionless lower free vibration frequency Ω_- (6.37), which is derived from the tolerance model versus ratio ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$, $E_2/E_1 = \{0.2, 0.8\}$, $d/\lambda = 0.1$.

In Figs. 6.6, 6.7 there are presented diagrams of dimensionless free vibration frequency Ω_+ (6.38), which is obtained in the framework of tolerance model versus ratio ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$, $E_2/E_1 = \{0.2, 0.8\}$, $d/\lambda = 0.1$.

In Figs. 6.8, 6.9 there are shown diagrams of dimensionless lower free vibration frequency Ω_- (6.37), versus ratio E_2/E_1, made for material properties distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$, $\rho_2/\rho_1 = \{0.2, 0.8\}$, $d/\lambda = 0.1$.

In Figs. 6.10, 6.11 there are shown diagrams of dimensionless higher free vibration frequency Ω_+ (6.38), versus ratio E_2/E_1, made for material properties distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$, $\rho_2/\rho_1 = \{0.2, 0.8\}$, $d/\lambda = 0.1$.

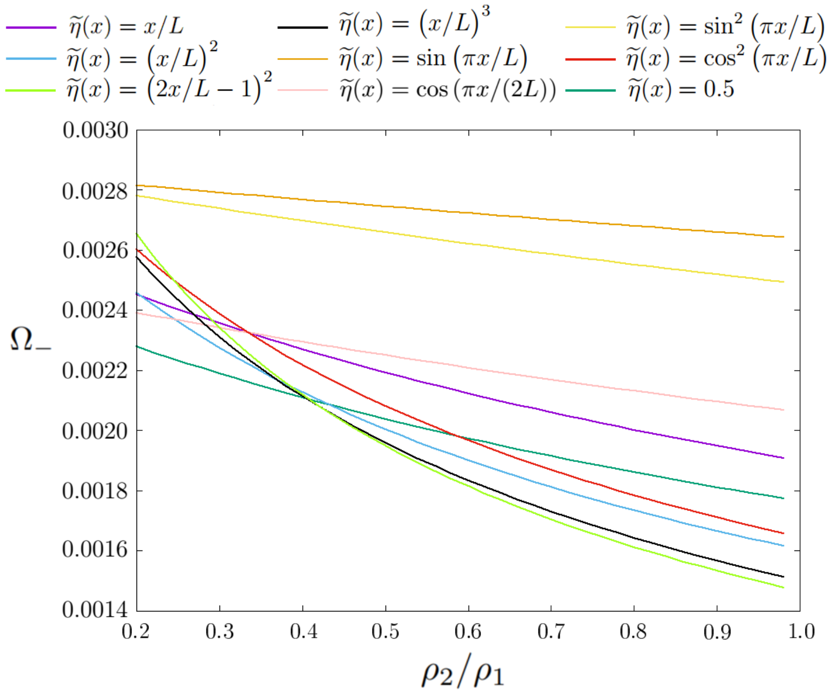

Figure 6.4: Diagrams of dimensionless lower free vibration frequency Ω_- (6.37) of the shell strip under consideration versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.2$, $\lambda/L = 0.1$

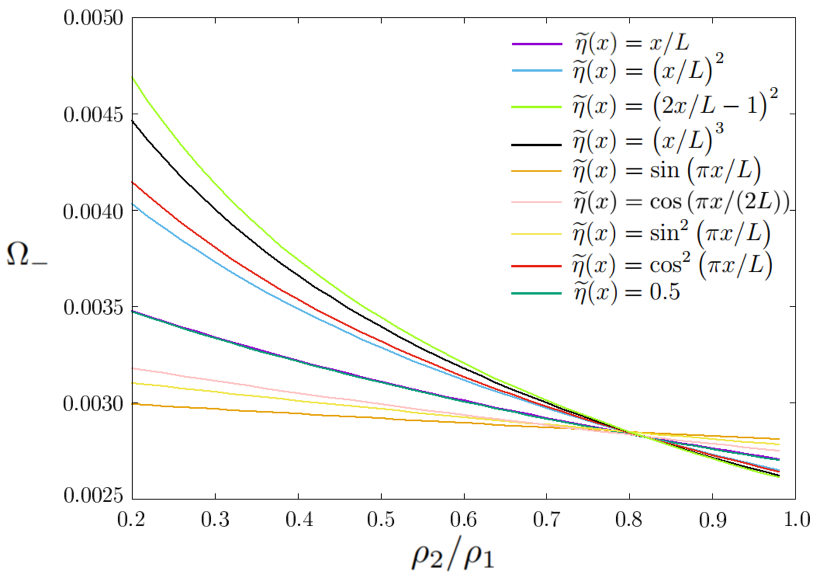

Figure 6.5: Diagrams of dimensionless lower free vibration frequency Ω_- (6.37) of of the shell strip under consideration versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.8$, $\lambda/L = 0.1$

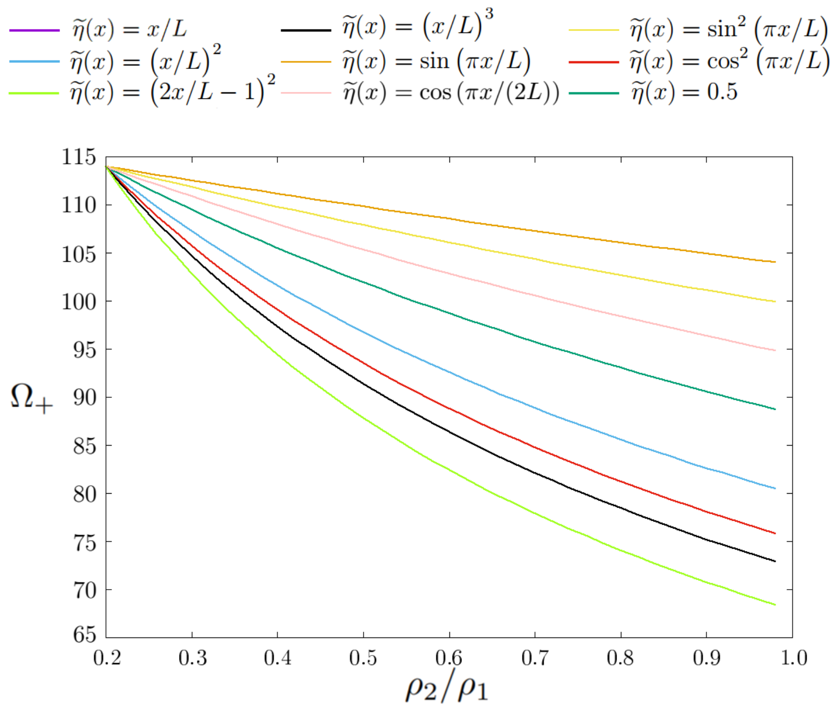

Figure 6.6: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.2$, $\lambda/L = 0.1$

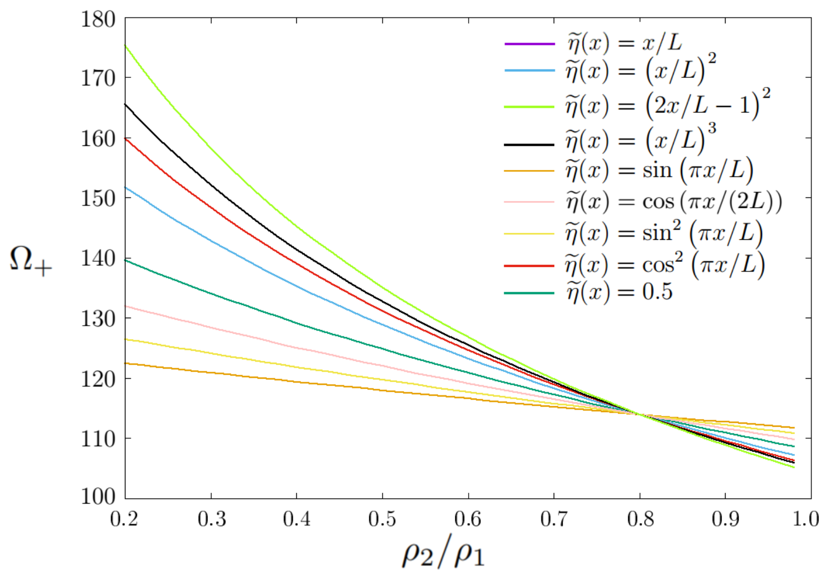

Figure 6.7: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.8$, $\lambda/L = 0.1$

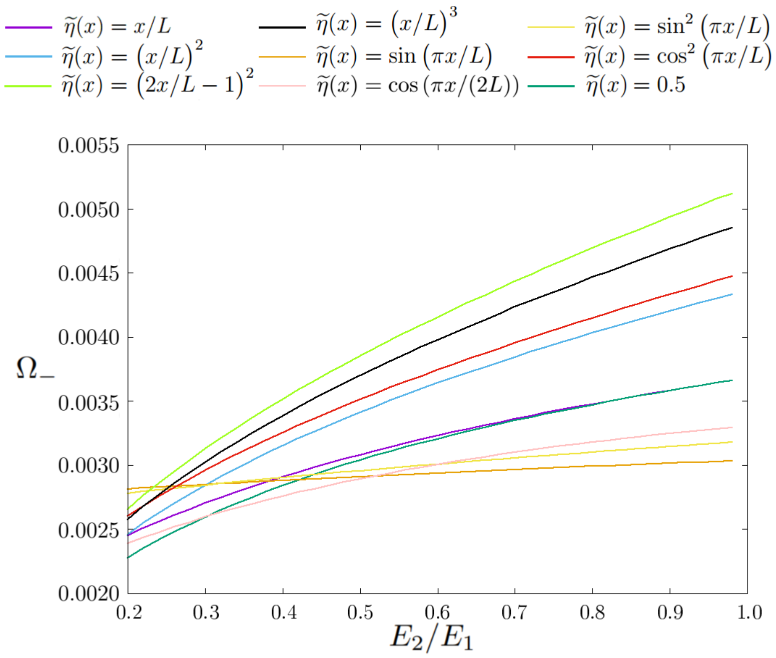

Figure 6.8: Diagrams of dimensionless lower free vibration frequency Ω_- (6.37) of the shell strip under consideration versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\rho_2/\rho_1 = 0.2$, $\lambda/L = 0.1$

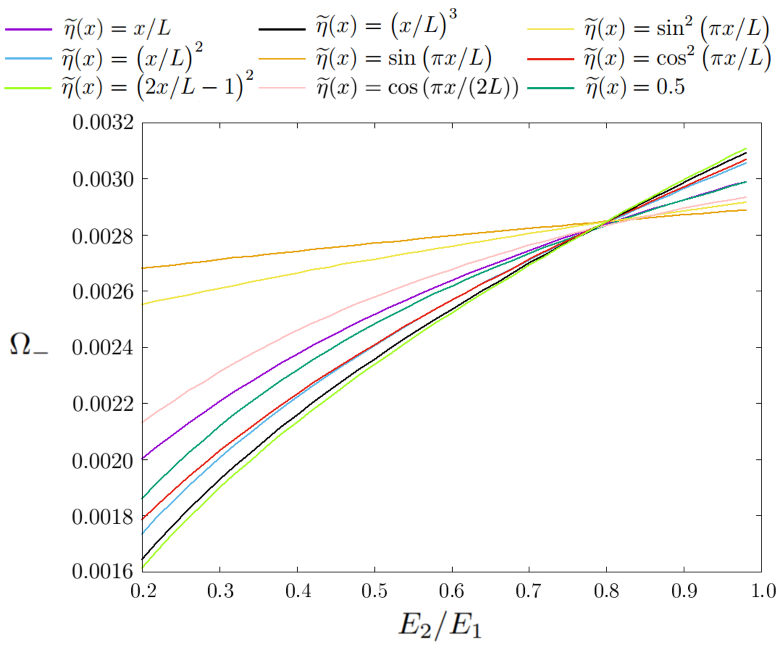

Figure 6.9: Diagrams of dimensionless lower free vibration frequency Ω_- (6.37) of the shell strip under consideration versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\rho_2/\rho_1 = 0.8$, $\lambda/L = 0.1$

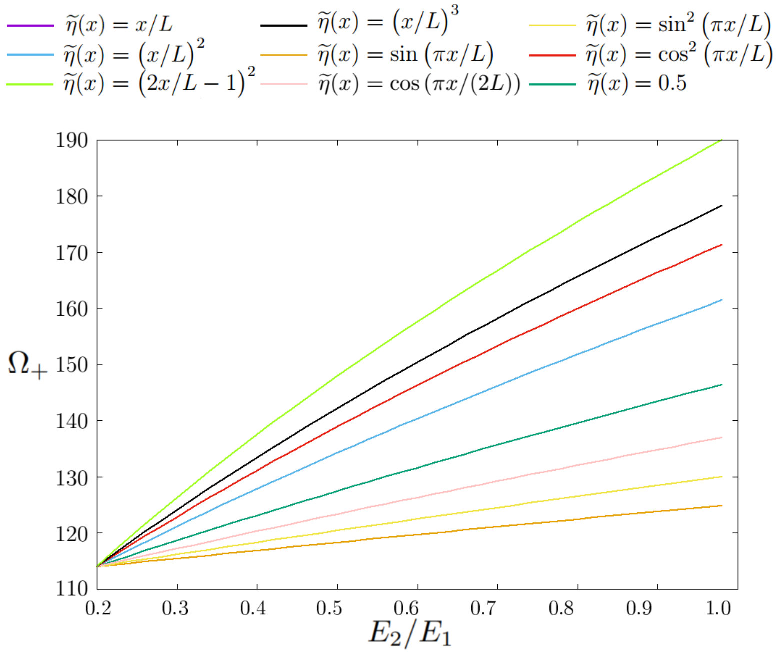

Figure 6.10: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\rho_2/\rho_1 = 0.2$, $\lambda/L = 0.1$

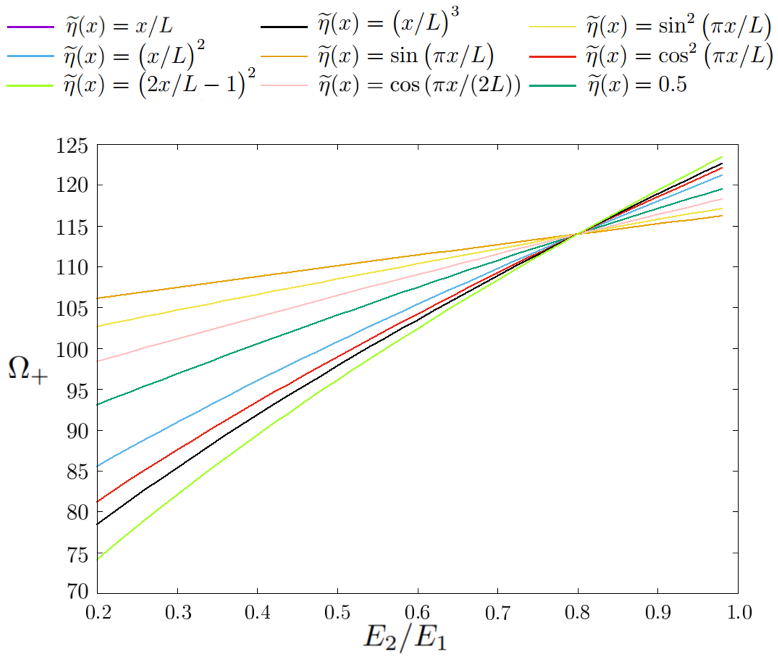

Figure 6.11: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\rho_2/\rho_1 = 0.8$, $\lambda/L = 0.1$

In Figs. 6.12 and 6.13 there are shown diagrams of dimensionless lower free vibration frequency Ω_- (6.37) from the tolerance model and of dimensionless free vibration frequency Ω^{AM} (6.39) from the asymptotic one, respectively, versus both ratios E_2/E_1 and ρ_2/ρ_1, made for material properties distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$, $d/\lambda = 0.1$.

From results shown in Figs. 6.12, 6.13, it follows that differences between Ω_- (6.37) and Ω^{AM} (6.39) are negligibly small (maximum relative error is equal to $1.56 \cdot 10^{-6}$). These numerical results coincide with analytical result (6.35). In the subsequent diagrams these frequencies will be taken into account simultaneously.

Plots of frequencies Ω_- (6.37), Ω^{AM} (6.39) versus both ratios E_2/E_1 and ρ_2/ρ_1 are presented in Figs. 6.14, 6.15. These plots are made for distribution functions of material properties $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $d/\lambda = 0.1$, $\lambda/L = \{0.01, 0.05\}$.

In Figs. 6.16-6.18 there are presented diagrams of dimensionless higher free vibration frequency Ω_+ (6.38), versus both ratios E_2/E_1 and ρ_2/ρ_1, made for material properties distribution functions $\widetilde{\eta}(x)$ defined by (6.6)-(6.13), (6.16) and for $d/\lambda = 0.1$, $\lambda/L = \{0.01, 0.05, 0.1\}$.

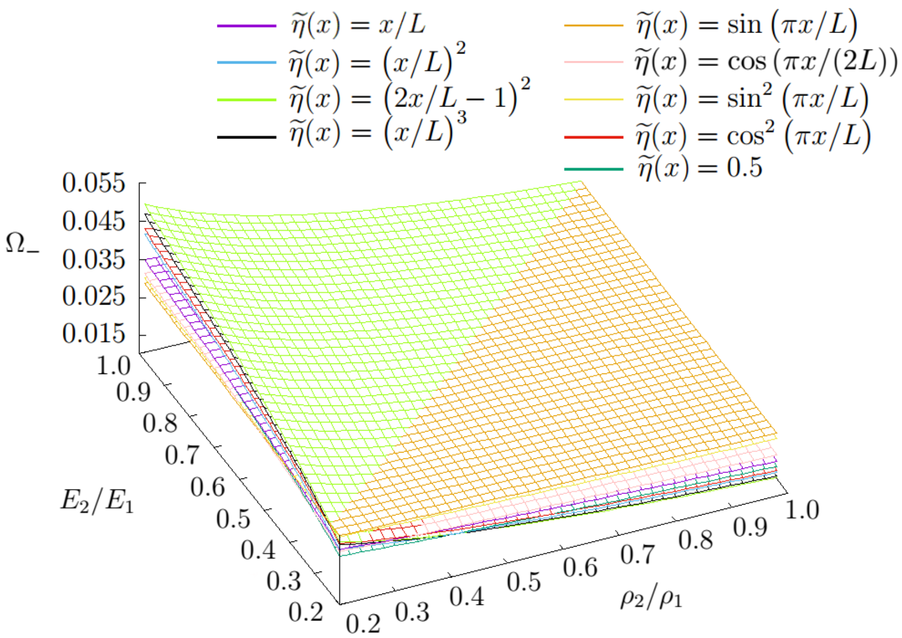

Figure 6.12: Diagrams of dimensionless lower free vibration frequency Ω_- (6.37) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$

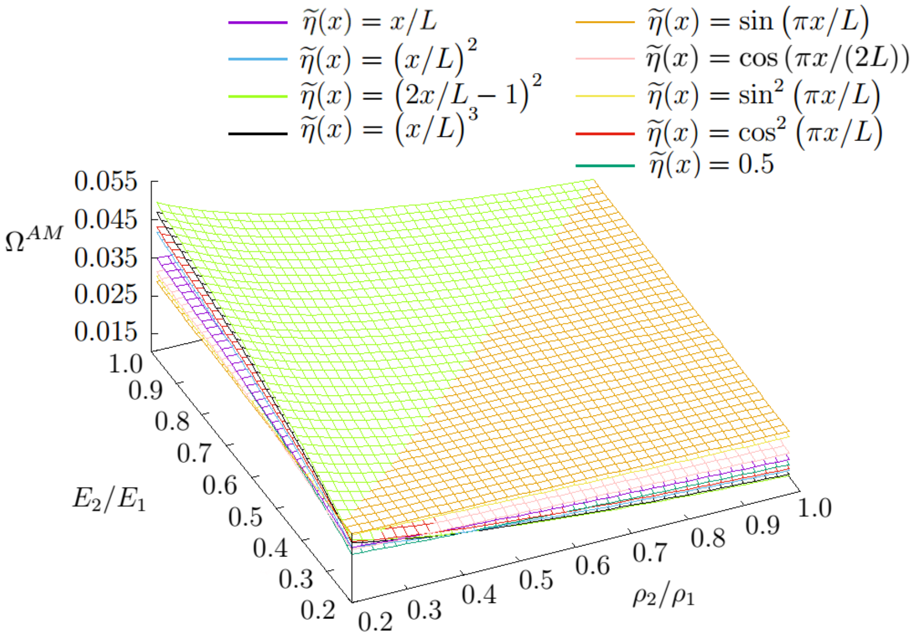

Figure 6.13: Diagrams of dimensionless lower free vibration frequency Ω^{AM} (6.39) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16), and for $\lambda/L = 0.1$

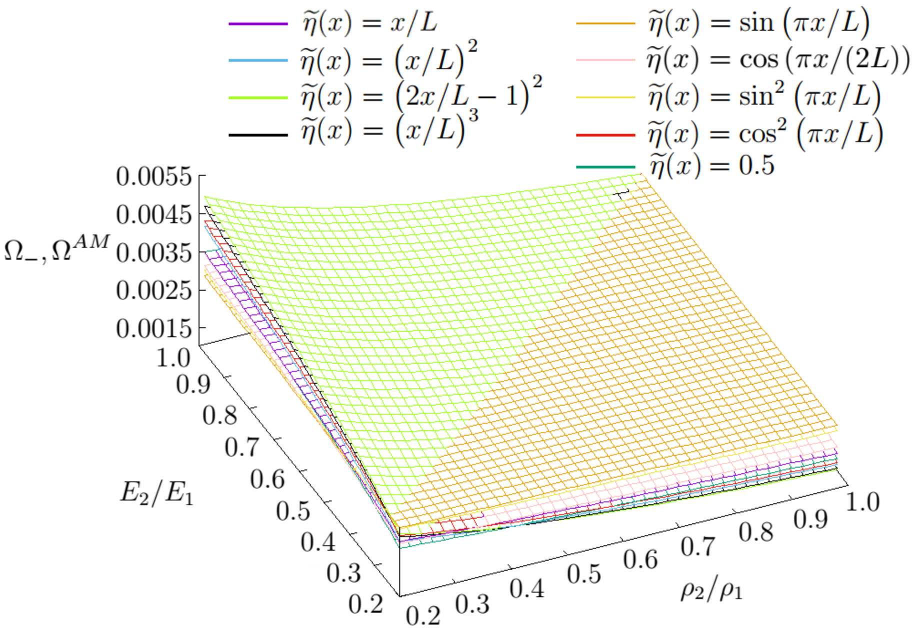

Figure 6.14: Diagrams of dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.01$

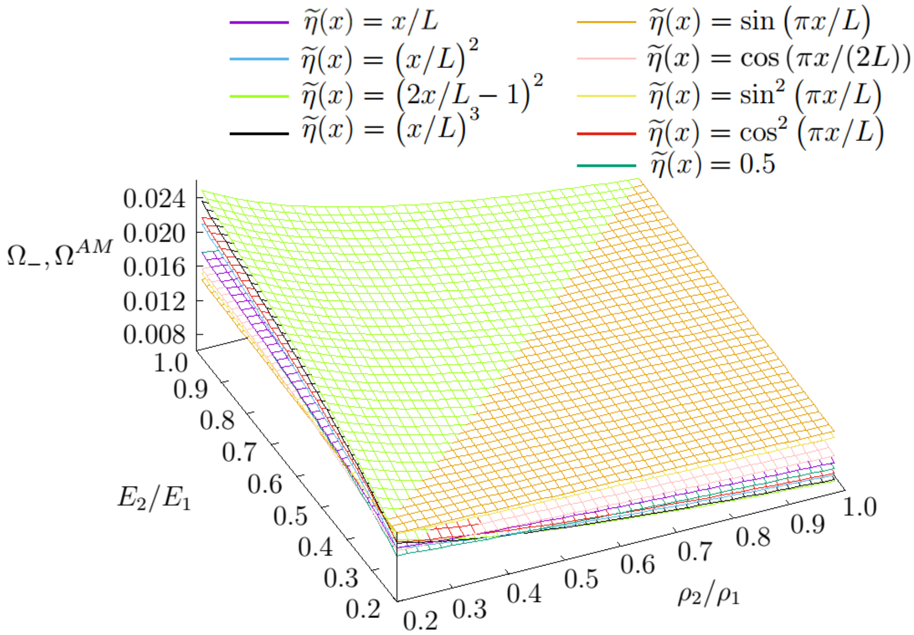

Figure 6.15: Diagrams of dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.05$

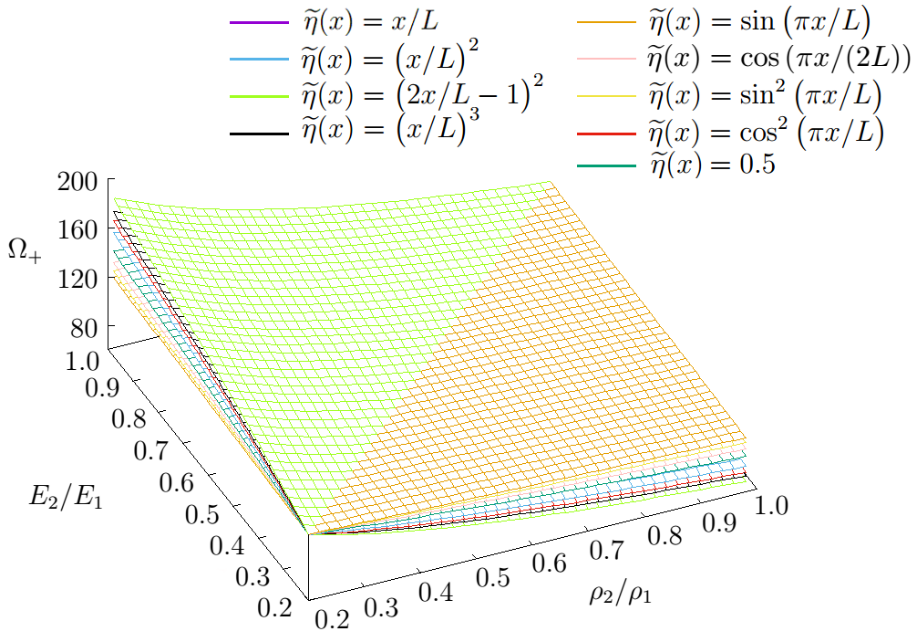

Figure 6.16: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.01$

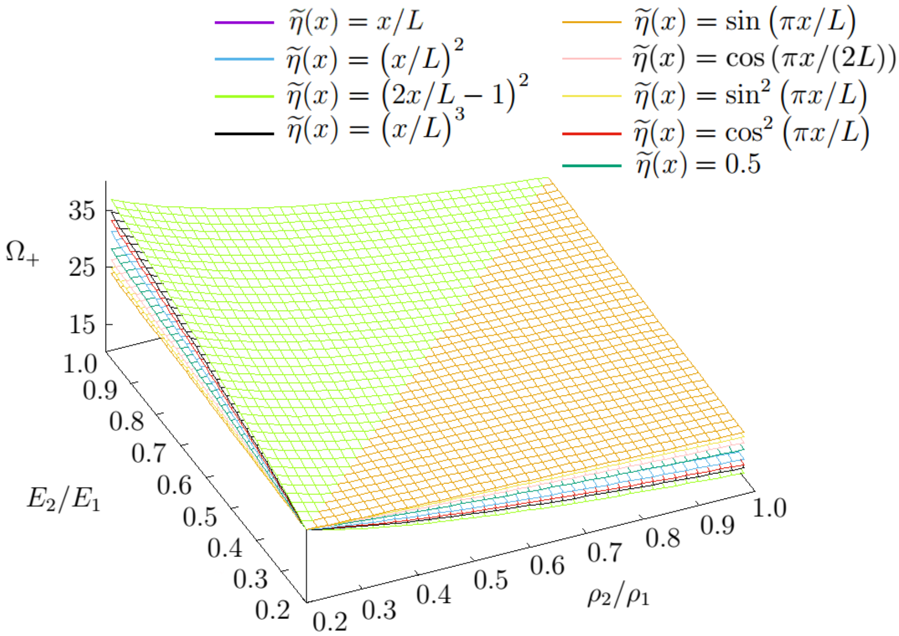

Figure 6.17: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.05$

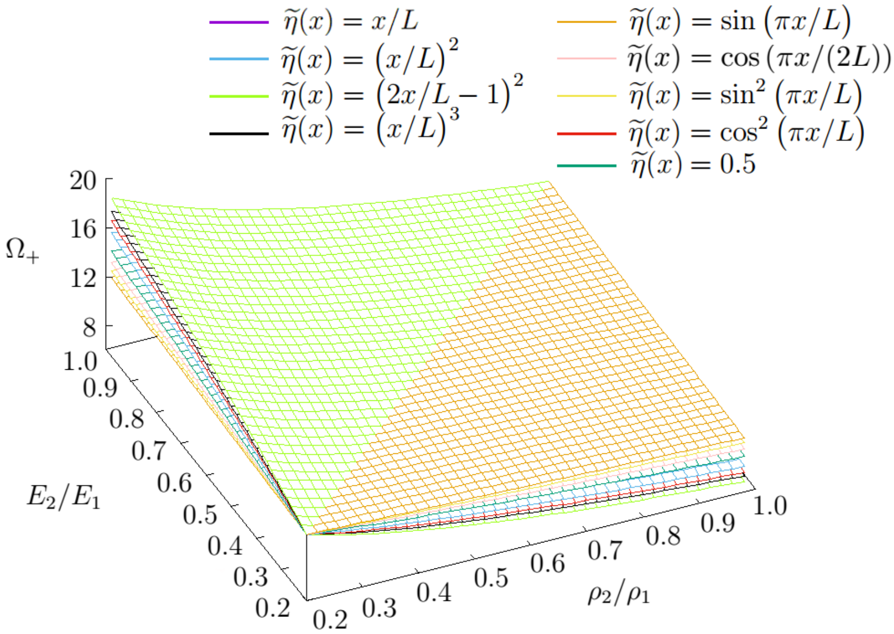

Figure 6.18: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $\lambda/L = 0.1$

In Figs. 6.19 and 6.20 there are shown diagrams of respectively dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) and dimensionless higher free vibration frequency Ω_+ (6.38), versus ratio $d/\lambda \in [0.01, 0.1]$, made for distribution functions of material properties described by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $\lambda/L = 0.1$.

In Figs. 6.21 and 6.22 there are shown diagrams of respectively dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) and dimensionless higher free vibration frequency Ω_+ (6.38), versus dimensionless microstructure length parameter $\lambda/L \in [0.01, 0.1]$, made for distribution functions of material properties described by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $d/\lambda = 0.1$.

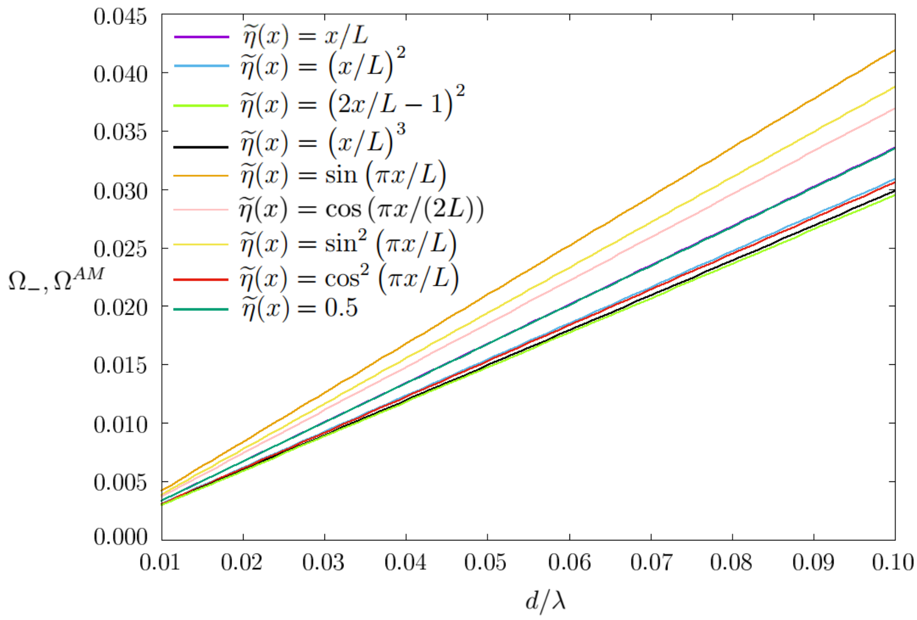

Figure 6.19: Diagrams of dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) of the shell strip under consideration versus ratio d/λ, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $\lambda/L = 0.1$

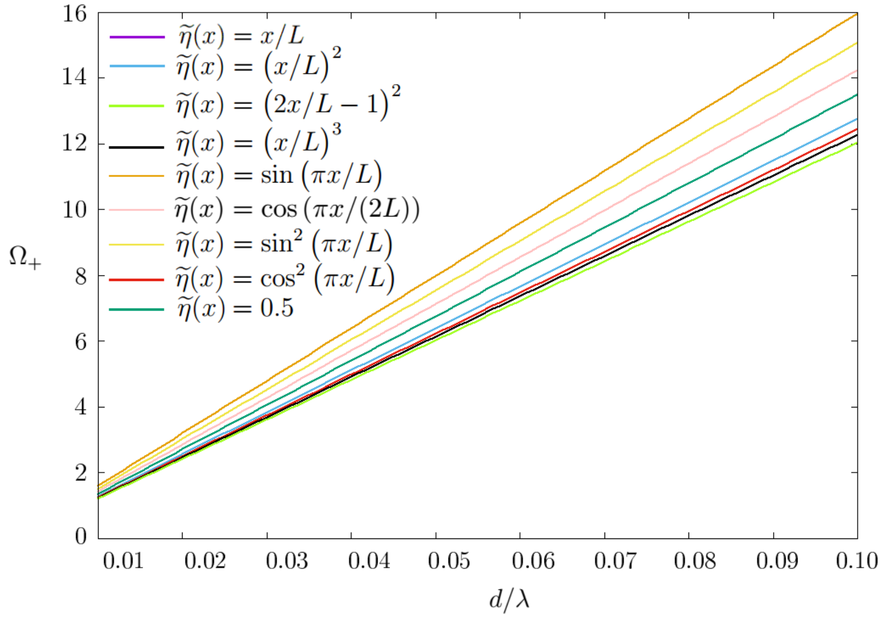

Figure 6.20: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus ratio d/λ, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $\lambda/L = 0.1$

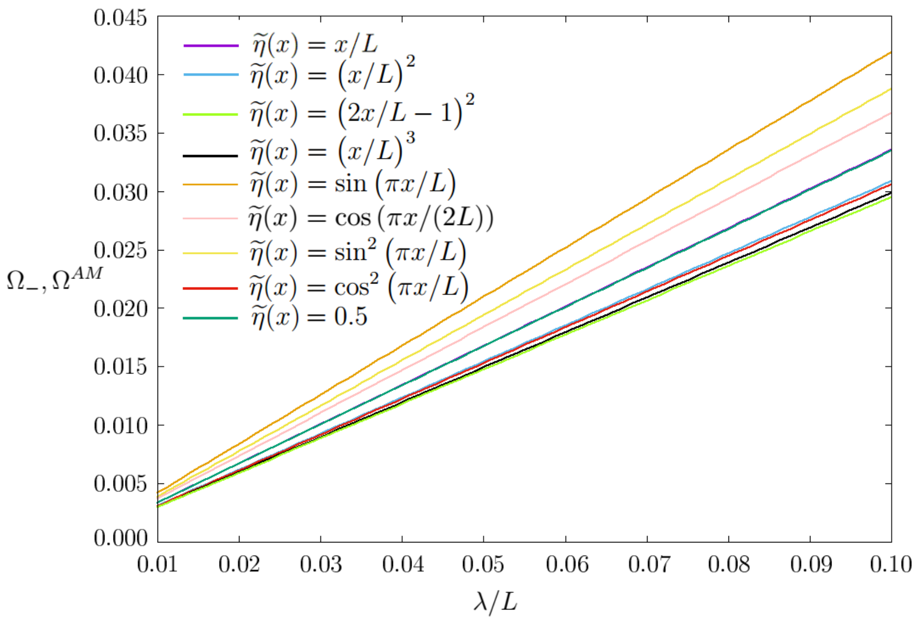

Figure 6.21: Diagrams of dimensionless lower free vibration frequencies Ω_- (6.37), Ω^{AM} (6.39) of the shell strip under consideration versus dimensionless microstructure length parameter λ/L, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $d/\lambda = 0.1$

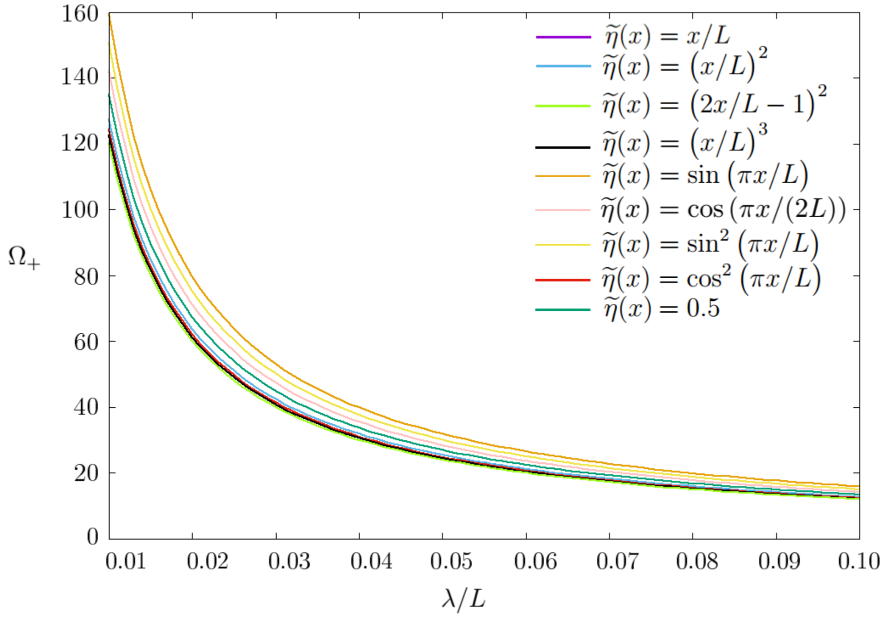

Figure 6.22: Diagrams of dimensionless higher free vibration frequency Ω_+ (6.38) of the shell strip under consideration versus dimensionless microstructure length parameter λ/L, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6)-(6.13), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$, $d/\lambda = 0.1$

6.2.6 Discussion of numerical results

On the basis results shown in Figs. 6.4-6.22 the following conclusions can be formulated:

- Agree with analytical results (6.35), values of dimensionless lower free vibration frequencies Ω_- and Ω^{AM} calculated in the framework of the tolerance and the asymptotic models are nearly identical, cf. Figs. 6.12 and 6.13.

- Values of dimensionless free vibration frequencies Ω_-, Ω_+, Ω^{AM} increase with the increasing of ratio $E_2/E_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between ***elastic properties*** of the shell component materials, cf. Figs. 6.8-6.18. Because the value of Young's module E_1 for the stronger material is fixed then these differences decrease if values of E_2 tend to value of E_1.

- Values of dimensionless free vibration frequencies Ω_-, Ω_+, Ω^{AM} decrease with the increasing of ratio $\rho_2/\rho_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between ***inertial properties*** of the shell component materials, cf. Figs. 6.4-6.7 and Figs. 6.12-6.18. Because the value of mass density ρ_1 for the stronger material is fixed then these differences decrease if values of ρ_2 tend to value of ρ_1.

- The highest values of frequencies Ω_-, Ω_+, Ω^{AM} are obtained for $\widetilde{\eta}(x) = \big(2x/L - 1\big)^2$ and for pair of ratios $(E_2/E_1 = 1.0,\ \rho_2/\rho_1 = 0.2)$, i.e. for a shell strip with a very strong inertial heterogeneity and with elastic homogeneous structure, cf. Figs. 6.12-6.18. The smallest values of Ω_-, Ω_+, Ω^{AM}, are obtained for $\widetilde{\eta}(x) = \big(2x/L - 1\big)^2$ and for pair of ratios $(E_2/E_1 = 0.2,\ \rho_2/\rho_1 = 1.0)$, i.e. for a shell strip with a very strong elastic heterogeneity and with inertial homogeneous structure, cf. Figs. 6.12-6.18.

- Values of the dimensionless frequencies Ω_-, Ω_+, Ω^{AM} increase linearly with the increasing of ratio d/λ, i.e. with the decreasing of differences between the shell thickness and the microstructure length parameter λ, cf. Figs. 6.19 and 6.20.

- Values of dimensionless lower free vibration frequencies Ω_-, Ω^{AM} increase linearly with the increasing of ratio λ/L, cf. Fig. 6.21. However, this increase is not caused by changes of λ with respect to L, but by changes of thickness d with respect to L. It follows from the fact that ratio $d/\lambda = 0.1$ is fixed and hence variations of values of $\varepsilon \equiv d/\lambda$ imply the changes of a shell thickness d with respect to $L =$ const., i.e. $d/\lambda = d/(\varepsilon L) = 0.1 \rightarrow d/L = 0.1\varepsilon$.

- Values of the dimensionless higher free vibration frequencies Ω_+ decrease with the increasing of ratio λ/L, i.e. with the decrease of differences between microstructure length parameter λ and the length dimension L of the shell midsurface in tolerant periodicity direction, cf. Fig. 6.22. For every distribution function $\widetilde{\eta}(x)$ under consideration, values of Ω_+ decrease very strongly for $\lambda/L \in [0.01, 0.03]$.

- From results shown in Fig. 6.6 it follows that for fixed value of ratio $E_2/E_1 = 0.2$ and for various values of ratio $\rho_2/\rho_1 \in [0.2, 1]$, the values of dimensionless higher free vibration frequency Ω_+ (6.38) obtained for $\widetilde{\eta}(x) = \cos(\pi x/(2L))$, $\widetilde{\eta}(x) = \sin(\pi x/L)$ and $\widetilde{\eta}(x) = \sin^2(\pi x/L)$ are always greater than for periodic shell strip, i.e. for distribution function $\widetilde{\eta}(x) = \eta = 0.5$. On the other hand, the values of dimensionless higher free vibration frequency Ω_+ (6.38) obtained for $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \cos^2(\pi x/L)$, $\widetilde{\eta}(x) = (x/L)^3$ $\widetilde{\eta}(x) = (2x/L - 1)^2$ are always smaller than for periodic shell strip.

- From results shown in Fig. 6.10 it follows that for fixed value of ratio $\rho_2/\rho_1 = 0.2$ and for various values of ratio $E_2/E_1 \in [0.2, 1]$, the values of dimensionless higher free vibration frequency Ω_+ (6.38) obtained for $\widetilde{\eta}(x) = \cos(\pi x/(2L))$, $\widetilde{\eta}(x) = \sin(\pi x/L)$ and $\widetilde{\eta}(x) = \sin^2(\pi x/L)$ are always smaller than for periodic shell strip, i.e. for distribution function $\widetilde{\eta}(x) = \eta = 0.5$. On the other hand, the values of dimensionless higher free vibration frequency Ω_+ (6.38) obtained for $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \cos^2(\pi x/L)$, $\widetilde{\eta}(x) = (x/L)^3$ and $\widetilde{\eta}(x) = (2x/L-1)^2$ are always greater than for periodic shell strip.

6.3. Free vibrations of the functionally graded open shell of finite length dimensions

6.3.1 Formulation of the problem

In this subsection transversal free vibrations of an open cylindrical shell with L_1, L_2, r, d as its circumferential length, axial length, midsurface curvature radius and constant thickness, respectively, are discussed. On the macroscopic level, the shell has a functionally graded material structure along circumferential direction. On the microscopic level, the shell is made of two elastic isotropic materials perfectly bonded on interfaces and densely, tolerance-periodically distributed along x-coordinate. The shell's structure in the axial direction is constant. Such a shell is shown in Fig. 6.1.

The basic cell defined by $\Delta \equiv [-\lambda/2, \lambda/2]$, cf. definition (6.1), is shown in Fig. 6.2.

Properties of the component materials are described by Young's moduli E_1, E_2, Poisson's ratio $\nu \equiv \nu_1 = \nu_2$ and mass densities ρ_1, ρ_2. Inside the cell, the elastic $E(x)$, $E \in TP^0_\delta(\Omega, \Delta)$ and inertial $\rho(x)$, $\rho \in TP^0_\delta(\Omega, \Delta)$, $x \in \Omega$ properties of the shell have periodic approximations $\widetilde{E}(x,z)$, $\widetilde{\rho}(x,z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$ defined by (6.2).

The rigidities $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x \in \Omega$ of the shell are described in Subsection 6.1.

The considerations will be based on equations (5.6), (5.7) of the tolerance model and equations (5.18) of the asymptotic model and restricted to the simplest forms of these models in which $a = n = A = N = 1$.

In order to investigate free vibrations, we assume that external forces f^α, f are equal to zero.

Moreover, the forces of inertia in directions tangential to the shell midsurface are neglected.

We also neglect fluctuating parts $h(x)U_\alpha(x,\xi,t)$ of displacements $u_\alpha(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$.

Periodic approximation $\widetilde{g}(x,z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, of fluctuation shape function $g(\cdot) \in FS^2_\delta(\Omega,\Delta)$ is given by (6.4).

The investigations will be carried out for a simply supported shell and for approximations $\widetilde{\eta}(x)$ of material properties distribution functions $\eta(x)$ given by (6.14) and (6.15), i.e. for $\widetilde{\eta}(x) = 0.6-0.2(2x/L-1)^2$ and $\widetilde{\eta}(x) = 0.6-0.2\sin(\pi x/L)$, respectively. We recall that $L \equiv L_1$.

Bearing in mind assumptions given above, the effect of a cell size on free vibration frequencies of the shell under consideration will be analysed by using both the tolerance model represented by equations of motion (5.7) with constitutive relations (5.6) and the asymptotic model governed by equations (5.18). Moreover, the influence of differences between elastic and inertial properties of the constituent materials on these frequencies will be studied.

The very important aim of this subsection is to verify the analytical results using numerical analysis performed with commercial computer software Ansys based on the finite element method.

6.3.2 Analysis in the framework of tolerance model

Now, the system of tolerance equations (5.7) reduces to the following system of four equations for $u^0_1(x,\xi,t)$, $u^0_2(x,\xi,t)$, $w^0(x,\xi,t)$, $W(x,\xi,t)$, $(x,\xi,t) \in \Omega\times\Xi\times\mathrm{I}$,

$$\begin{aligned}
&\partial_\alpha\left(\left\langle D^{\alpha\beta\gamma\delta}\right\rangle\partial_\delta u^0_\gamma + r^{-1}\left\langle D^{\alpha\beta 11}\right\rangle w^0\right) = 0,\\
&\partial_{\alpha\beta}\left(\left\langle B^{\alpha\beta\gamma\delta}\right\rangle\partial_{\gamma\delta}w^0 + \left\langle B^{\alpha\beta 11}\partial_{11}g\right\rangle W + \lambda^2\left\langle B^{\alpha\beta 22}\overline{g}\right\rangle\partial_{22}W\right) +\\
&+ r^{-1}\left\langle D^{11\gamma\delta}\right\rangle\partial_\delta u^0_\gamma + r^{-2}\left\langle D^{1111}\right\rangle w^0 + \langle\mu\rangle\,\ddot{w}^0 = 0,\\
&\left\langle B^{11\alpha\beta}\partial_{11}g\right\rangle\partial_{\alpha\beta}w^0 + \lambda^2\left\langle B^{\alpha\beta 22}\overline{g}\right\rangle\partial_{\alpha\beta 22}w^0 + \left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle W+\\
&+ 2\lambda^2\left\langle\overline{g}B^{1122}\partial_{11}g\right\rangle\partial_{22}W - 4\lambda^2\left\langle B^{1212}\left(\partial_1\widehat{g}\right)^2\right\rangle\partial_{22}W+\\
&+ \lambda^4\left\langle B^{2222}\overline{g}^2\right\rangle\partial_{2222}W + \lambda^4\left\langle\mu\overline{g}^2\right\rangle\ddot{W} = 0,
\end{aligned}\tag{6.40}$$

where $\widehat{g}(\cdot) = \lambda^{-1}g(\cdot)$, $\overline{g}(\cdot) = \lambda^{-2}g(\cdot)$. We recall that derivative $\partial_{11}g(x)$ of fluctuation shape function $g(x)$ is independent of λ as parameter. In equations (6.40) some terms depend on microstructure length parameter λ. All coefficients of (6.40) are continuous and slowly-varying functions in argument x.

It is difficult to find analytical solutions to Eqs. (6.40). Thus, similarly as Subsection 6.2, in order to obtain approximate formulas of free vibration frequencies the known Ritz method can be applied, cf. Kaliski [46]. Using this method, formulas of the maximal strain energy $E_{\max}$ and the maximal kinetic energy $K_{\max}$ are determined.

In the problem under consideration, strain energy function $E = E(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, related to the shell midsurface has the form given by (4.2). Below, we recall this function

$$E = \frac{1}{2}\left(D^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + B^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta}\right), \tag{6.41}$$

where $\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta) - b_{\alpha\beta}w$, $\kappa_{\alpha\beta} = -\partial_{\alpha\beta}w$, $b_{11} = -r^{-1}$, $b_{22} = b_{12} = b_{21} = 0$.

The kinetic energy function $K = K(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, related to the shell midsurface is given by

$$K = \frac{1}{2}\mu\left(\dot{w}\right)^2. \tag{6.42}$$

Tolerance modelling applied to (6.41) and (6.42) yields the results

$$\begin{aligned}
\langle E\rangle(x) = \frac{1}{2}\Bigg[&\left\langle D^{1111}\right\rangle(x)\left(\left(\partial_1 u_1^0\right)^2 + \frac{2}{r}\partial_1 u_1^0 w^0 + \frac{1}{r^2}\left(w^0\right)^2\right) + \\
&+ \left\langle D^{2222}\right\rangle(x)\left(\partial_2 u_2^0\right)^2 + 2\left\langle D^{1122}\right\rangle(x)\left(\partial_1 u_1^0\partial_2 u_2^0 + \frac{1}{r}w^0\partial_2 u_2^0\right) + \\
&+ \left\langle D^{1212}\right\rangle(x)\left(\left(\partial_2 u_1^0\right)^2 + \left(\partial_1 u_2^0\right)^2 + 2\partial_2 u_1^0\partial_1 u_2^0\right) + \left\langle B^{1111}\right\rangle(x)\left(\partial_{11}w^0\right)^2 + \\
&+ 2\left\langle B^{1111}\partial_{11}g\right\rangle(x)\partial_{11}w^0 W + \left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle(x)W^2 + \\
&+ 2\Big(\left\langle B^{1122}\right\rangle(x)\partial_{11}w^0\partial_{22}w^0 + \lambda^2\left\langle B^{1122}\overline{g}\right\rangle(x)\partial_{11}w^0\partial_{22}W + \\
&+ \left\langle B^{1122}\partial_{11}g\right\rangle(x)\partial_{22}w^0 W + \lambda^2\left\langle B^{1122}\overline{g}\partial_{11}g\right\rangle(x)W\partial_{22}W\Big) + \\
&+ \left\langle B^{2222}\right\rangle(x)\left(\partial_{22}w^0\right)^2 + 2\lambda^2\left\langle B^{2222}\overline{g}\right\rangle(x)\partial_{22}w^0\partial_{22}W + \\
&+ \lambda^4\left\langle B^{2222}\overline{g}^2\right\rangle(x)\left(\partial_{22}W\right)^2 + 4\left\langle B^{1212}\right\rangle(x)\left(\partial_{12}w^0\right)^2 + \\
&+ 4\lambda^2\left\langle B^{1212}\left(\partial_1\widehat{g}\right)^2\right\rangle(x)\left(\partial_2 W\right)^2\Bigg], \quad x \in \Omega_\Delta,
\end{aligned} \tag{6.43}$$

and

$$\langle K\rangle(x) = \frac{1}{2}\left[\langle\mu\rangle(x)\left(\dot{w}^0\right)^2 + \lambda^4\left\langle\mu\left(\overline{g}\right)^2\right\rangle(x)\left(\dot{W}\right)^2\right], \quad x \in \Omega_\Delta. \tag{6.44}$$

The maximal strain energy and the maximal kinetic energy will be obtained by integrating results (6.43) and (6.44) over region $\Omega \times \Xi = (0, L_1) \times (0, L_2)$. Unknown functions in (6.43), (6.44) must satisfy the given boundary conditions for $x = 0$, $x = L_1$ and $\xi = 0$, $\xi = L_2$.

Solutions to Eqs. (6.40) satisfying the boundary conditions for a simply supported shell can be assumed in the form

$$
\begin{aligned}
u_1^0(x,\xi,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\left(\alpha_m x\right)\sin\left(\beta_n \xi\right)\cos\left(\widehat{\omega}_{mn} t\right),\\
u_2^0(x,\xi,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\left(\alpha_m x\right)\cos\left(\beta_n \xi\right)\cos\left(\widehat{\omega}_{mn} t\right),\\
w^0(x,\xi,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\left(\alpha_m x\right)\sin\left(\beta_n \xi\right)\cos\left(\widehat{\omega}_{mn} t\right),\\
W(x,\xi,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin\left(\alpha_m x\right)\sin\left(\beta_n \xi\right)\cos\left(\widehat{\omega}_{mn} t\right),
\end{aligned}
\tag{6.45}
$$

where $\alpha = m\pi/L_1$ and $\beta = n\pi/L_2$ are wave numbers, $\widehat{\omega}_{mn}$ is a frequency of transverse free vibrations.

Let us introduce the following denotations

$$
\begin{aligned}
\overline{a}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle D^{1111}\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,\\
\overline{b}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle D^{2222}\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,\\
\overline{d}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle D^{1122}\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,\\
\widehat{d}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle D^{1212}\right\rangle \cos^2\left(\alpha_m x\right)\cos^2\left(\beta_n \xi\right) d\xi dx,\\
\widetilde{e}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1111}\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,\\
\overline{e}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1111}\partial_{11} g\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,\\
\widehat{e}_{mn} &\equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1111}\left(\partial_{11} g\right)^2\right\rangle \sin^2\left(\alpha_m x\right)\sin^2\left(\beta_n \xi\right) d\xi dx,
\end{aligned}
\tag{6.46}
$$

$$\widetilde{k}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1122} \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$\overline{k}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1122}\overline{g} \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$\widehat{k}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1122}\partial_{11} g \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$p_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1122}\overline{g}\partial_{11} g \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$\widetilde{p}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{2222} \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$\overline{p}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{2222}\overline{g} \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx, \qquad (6.46_{contd})$$

$$\widehat{p}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{2222}\overline{g}^2 \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$s_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1212} \right\rangle \cos^2(\alpha_m x)\cos^2(\beta_n \xi)\, d\xi dx,$$

$$\widetilde{s}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle B^{1212}\left(\partial_1 \widehat{g}\right)^2 \right\rangle \sin^2(\alpha_m x)\cos^2(\beta_n \xi)\, d\xi dx,$$

$$\overline{s}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle \mu \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

$$\widehat{s}_{mn} \equiv \int_0^{L_1}\int_0^{L_2} \left\langle \mu\left(\overline{g}\right)^2 \right\rangle \sin^2(\alpha_m x)\sin^2(\beta_n \xi)\, d\xi dx,$$

where averages $\langle \cdot \rangle$ are given in Appendix, cf. (A.1)-(A.19) and (A.21). We recall that $\widehat{g} = \lambda^{-1} g(\cdot)$, $\overline{g}(\cdot) = \lambda^{-2} g(\cdot)$.

Taking into account (6.45) and using denotations (6.46), for arbitrary but fixed m, n, the maximal strain energy $E_{\max}$ and the maximal kinetic energy $K_{\max}$ by the tolerance model can be written as

$$
\begin{aligned}
E_{\max} = \frac{1}{2}\Bigg[& A_{mn}^2\left(\alpha_m^2\overline{a}_m + \beta_n^2\,\widehat{d}_{mn}\right) + B_{mn}^2\left(\beta_n^2\overline{b}_{mn} + \alpha_m^2\widehat{d}_{mn}\right) + \\
& + C_{mn}^2\left(r^{-2}\overline{a}_{mn} + \alpha_m^4\widetilde{e}_{mn} + 2\alpha_m^2\beta_n^2\left(\widetilde{k}_{mn} + 2s_{mn}\right) + \beta_n^4\widetilde{p}_{mn}\right) + \\
& + D_{mn}^2\left(\widehat{e}_{mn} + 2\lambda^2\beta_n^2\left(2\widetilde{s}_{mn} - p_{mn}\right) + \lambda^4\beta_n^4\,\widehat{p}_{mn}\right) + \\
& + 2A_{mn}B_{mn}\alpha_m\beta_n\left(\overline{d}_{mn} - \widehat{d}_{mn}\right) - 2A_{mn}C_{mn}r^{-1}\alpha_m\overline{a}_{mn} + \\
& - 2B_{mn}C_{mn}r^{-1}\beta_n\overline{d}_{mn} + \\
& + 2C_{mn}D_{mn}\left(\lambda^2\left(\alpha_m^2\beta_n^2\,\overline{k}_{mn} + \beta_n^4\,\overline{p}_{mn}\right) - \left(\alpha_m^2\overline{e}_{mn} + \beta_n^2\widehat{k}_{mn}\right)\right)\Bigg],
\end{aligned} \tag{6.47}
$$

$$
K_{\max} = \frac{1}{2}\left[C_{mn}^2\overline{s}_{mn} + D_{mn}^2\lambda^4\,\widehat{s}_{mn}\right]\widehat{\omega}_{mn}^2. \tag{6.48}
$$

Substituting the right-hand sides of (6.47) and (6.48) into the conditions of the Ritz method

$$
\begin{aligned}
\frac{\partial\left(E_{\max} - K_{\max}\right)}{\partial A_{mn}} &= 0, \\
\frac{\partial\left(E_{\max} - K_{\max}\right)}{\partial B_{mn}} &= 0, \\
\frac{\partial\left(E_{\max} - K_{\max}\right)}{\partial C_{mn}} &= 0, \\
\frac{\partial\left(E_{\max} - K_{\max}\right)}{\partial D_{mn}} &= 0,
\end{aligned} \tag{6.49}
$$

we obtain from (6.49), for arbitrary but fixed m, n, the system of four linear homogeneous algebraic equations for A_{mn}, B_{mn}, C_{mn}, D_{mn}. For a non-trivial solution, the determinant of this system must be equal to zero. In this manner we arrive at the characteristic equation for frequency $\widehat{\omega}_{mn}$ of the transverse free vibrations of the shell under consideration. Using (6.46) we introduce the extra denotations

$$
\begin{aligned}
& P_{mn} \equiv \alpha_m^2\overline{a}_{mn} + \beta_n^2\,\widehat{d}_{mn}, \quad \overline{P}_{mn} \equiv \beta_n^2\overline{b}_{mn} + \alpha_m^2\,\widehat{d}_{mn}, \\
& \widehat{P}_{mn} \equiv r^{-2}\overline{a}_{mn} + \alpha_m^4\widetilde{e}_{mn} + 2\alpha_m^2\beta_n^2\left(\overline{k}_{mn} + 2s_{mn}\right) + \beta_n^4\widetilde{p}_{mn}, \\
& \widehat{R}_{mn} \equiv \alpha_m\beta_n\left(\overline{d}_{mn} - \widehat{d}_{mn}\right), \quad S_{mn} \equiv r^{-1}\alpha_m\overline{a}_{mn}, \quad \overline{S}_{mn} \equiv r^{-1}\beta_n\overline{d}_{mn}, \\
& \widetilde{R}_{mn} \equiv 2\beta_n^2\left(2\widetilde{s}_{mn} - p_{mn}\right), \quad \widetilde{\widetilde{R}}_{mn} \equiv \beta_n^4\,\widehat{p}_{mn}, \\
& \widetilde{S}_{mn} \equiv \alpha_m^2\beta_n^2\overline{k}_{mn} + \beta_n^4\,\overline{p}, \quad \widetilde{\widetilde{S}}_{mn} \equiv \alpha_m^2\overline{e}_{mn} + \beta_n^2\widehat{k}_{mn},
\end{aligned} \tag{6.50}
$$

and

$$
\begin{aligned}
\chi_{mn} \equiv \widehat{P}_{mn} - \frac{S_{mn}\widehat{R}_{mn}\left(\widehat{R}_{mn}S_{mn} - P_{mn}\overline{S}_{mn}\right)}{P_{mn}\left((\overline{P}_{mn})^2 - (\widehat{R}_{mn})^2\right)} - \frac{(S_{mn})^2}{P_{mn}} + \\
+ \frac{\overline{S}_{mn}\left(\widehat{R}_{mn}S_{mn} - P_{mn}\overline{S}_{mn}\right)}{(P_{mn})^2 - (\widehat{R}_{mn})^2}.
\end{aligned}
\tag{6.51}
$$

Under denotations $\widehat{e}_{mn}$, $\overline{s}_{mn}$, $\widehat{s}_{mn}$ given by $(6.46)_7$, $(6.46)_{17,18}$, respectively, and taking into account expression (6.51) for χ_{mn} together with $(6.50)_{1-6}$ as well as notations $\widetilde{R}_{mn}$, $\widetilde{\widetilde{R}}_{mn}$, $\widetilde{S}_{mn}$, $\widetilde{\widetilde{S}}_{mn}$ given by $(6.50)_{7-10}$, from the characteristic equation mentioned above we derive the following formulae for *the lower free vibration frequency* $\widehat{\omega}_{mn-}$ and for *the new additional higher free vibration frequency* $\widehat{\omega}_{mn+}$, caused by a tolerance-periodic structure of the shell

$$
\begin{aligned}
(\widehat{\omega}_{mn-})^2 = \frac{1}{2}\left(\frac{\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}}{\lambda^4\,\widehat{s}_{mn}} + \frac{\chi_{mn}}{\overline{s}_{mn}}\right) + \\
- \frac{1}{2\lambda^4\,\widehat{s}_{mn}\overline{s}_{mn}}\left[\left(\left(\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}\right)\overline{s}_{mn} + \lambda^4\,\widehat{s}_{mn}\chi_{mn}\right)^2 + \right. \\
- 4\lambda^4\,\widehat{s}_{mn}\overline{s}_{mn}\left(\left(\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}\right)\chi_{mn} + \right. \\
\left.\left. - \left(\lambda^2\widetilde{S}_{mn} - \widetilde{\widetilde{S}}_{mn}\right)^2\right)\right]^{1/2},
\end{aligned}
\tag{6.52}
$$

$$
\begin{aligned}
(\widehat{\omega}_{mn+})^2 = \frac{1}{2}\left(\frac{\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}}{\lambda^4\,\widehat{s}_{mn}} + \frac{\chi_{mn}}{\overline{s}_{mn}}\right) + \\
+ \frac{1}{2\lambda^4\,\widehat{s}_{mn}\overline{s}_{mn}}\left[\left(\left(\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}\right)\overline{s}_{mn} + \lambda^4\,\widehat{s}_{mn}\chi_{mn}\right)^2 + \right. \\
- 4\lambda^4\,\widehat{s}_{mn}\overline{s}_{mn}\left(\left(\widehat{e}_{mn} + \lambda^2\widetilde{R}_{mn} + \lambda^4\widetilde{\widetilde{R}}_{mn}\right)\chi_{mn} + \right. \\
\left.\left. - \left(\lambda^2\widetilde{S}_{mn} - \widetilde{\widetilde{S}}_{mn}\right)^2\right)\right]^{1/2}.
\end{aligned}
\tag{6.53}
$$

Results (6.52), (6.53) depend explicitly on a microstructure length parameter λ.

6.3.3 Analysis in the framework of asymptotic model

In order to evaluate the obtained results, let us consider the above problem within the asymptotic model. Now, asymptotic model equations (5.18) reduce to the following form

$$
\begin{aligned}
&\partial_\alpha \left(\left\langle D^{\alpha\beta\gamma\delta} \right\rangle \partial_\delta u^0_\gamma + r^{-1} \left\langle D^{\alpha\beta 11} \right\rangle w^0 \right) = 0, \\
&\partial_{\alpha\beta} \left(\left\langle B^{\alpha\beta\gamma\delta} \right\rangle \partial_{\gamma\delta} w^0 + \left\langle B^{\alpha\beta 11} \partial_{11} g \right\rangle W \right) + \\
&+ r^{-1} \left\langle D^{11\gamma\delta} \right\rangle \partial_\delta u^0_\gamma + r^{-2} \left\langle D^{1111} \right\rangle w^0 + \langle \mu \rangle \ddot{w}^0 = 0, \\
&\left\langle B^{11\alpha\beta} \partial_{11} g \right\rangle \partial_{\alpha\beta} w^0 + \left\langle B^{1111} \left(\partial_{11} g \right)^2 \right\rangle W = 0,
\end{aligned}
\tag{6.54}
$$

Note, that (6.54) can also be derived directly from governing equations (6.40) by neglecting terms depending explicitly on microstructure length parameter λ.

Asymptotic modelling applied to strain energy (6.41) yields the result having form of (6.43) without terms depending on microstructure length parameter λ.

Asymptotic modelling applied to kinetic energy (6.42) leads to result having form of (6.44) without a term depending on a cell size λ.

Assuming solutions to (6.54) in the form of (6.45) and using denotations (6.46), for arbitrary but fixed m, n, we derive formulas for the maximal strain energy and the maximal kinetic energy. The maximal strain energy obtained in the framework of asymptotic model has a form of result (6.47) derived within the tolerance model but without terms depending on λ. The maximal kinetic energy has the form of result (6.48) obtained within the tolerance model but without a term including λ and with free vibration frequency $\widehat{\omega}^{AM}_{mn}$ of the asymptotic model which replaced free vibration frequency $\widehat{\omega}_{mn}$ of the tolerance one.

Neglecting in (6.47), (6.48) the length-scale terms and replacing in (6.48) frequency $\widehat{\omega}_{mn}$ by $\widehat{\omega}^{AM}_{mn}$ and then substituting the right-hand sides of (6.47), (6.48) into conditions (6.49) of the Ritz method, after some manipulations the formula for frequency $\widehat{\omega}^{AM}_{mn}$ of the shell's transverse free vibrations is derived in the framework of the asymptotic model under consideration

$$
\left(\widehat{\omega}^{AM}_{mn} \right)^2 = \frac{\chi_{mn}}{\overline{s}_{mn}} - \frac{(\widetilde{\widetilde{S}}_{mn})^2}{\widehat{e}_{mn} \overline{s}_{mn}},
\tag{6.55}
$$

where $\widehat{e}_{mn}$, $\overline{s}_{mn}$, $\widetilde{\widetilde{S}}_{mn}$, χ_{mn} are given by $(6.46)_7$, $(6.46)_{17}$, $(6.50)_{10}$, (6.51), respectively. ***This frequency is independent of a cell size.***

6.3.4 Numerical calculations

For the simply supported shell under consideration, calculations are made for approximations $\widetilde{\eta}(x)$ of material properties distribution functions $\eta(x)$ given by (6.14)-(6.16), i.e. for $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$, $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and $\widetilde{\eta}(x) = \eta = 0.5$.

Diagrams of these functions are shown in Fig. 6.3.

Calculations are made for Poisson's ratio $\nu = 0.3$, for fixed ratios $L_2/L_1 = 2/\pi$ (short shell), $d/\lambda = 12/(125\pi)$, $\lambda/L_1 = 1/24$ and for various ratios $E_2/E_1 \in [0.2, 1]$, $\rho_2/\rho_1 \in [0.2, 1]$.

From numerical analysis carried out it follows that for the shell under consideration, the smallest free vibration frequency related to the lowest free vibration mode is obtained for $m = 5$ and $n = 1$, i.e. for wave numbers $\alpha = 5\pi/L_1$ and $\beta = \pi/L_2$.

In the subsequent considerations, by $\widehat{\omega}_-$, $\widehat{\omega}_+$, $\widehat{\omega}^{AM}$ we shall denote frequencies related to $m = 5$ and $n = 1$.

We define the following dimensionless free vibration frequencies

$$\left(\widehat{\Omega}_-\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\widehat{\omega}_-\right)^2, \tag{6.56}$$

$$\left(\widehat{\Omega}_+\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\widehat{\omega}_+\right)^2, \tag{6.57}$$

$$\left(\widehat{\Omega}^{AM}\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1}\left(\widehat{\omega}^{AM}\right)^2, \tag{6.58}$$

where $\widehat{\omega}_-$, $\widehat{\omega}_+$, $\widehat{\omega}^{AM}$ are given by (6.52), (6.53), (6.55), respectively. We recall that $L \equiv L_1$.

Results of calculations are given in Figs. 6.23 and 6.24.

In Fig. 6.23 there are presented diagrams of lower free dimensionless vibration frequencies $\widehat{\Omega}_-$ (6.56), $\widehat{\Omega}^{AM}$ (6.58) derived from the tolerance and asymptotic models, respectively, versus both ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ described by (6.14)-(6.16) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$.

In Fig. 6.24 there are shown diagrams of higher dimensionless free vibration frequency $\widehat{\Omega}_+$ (6.57) derived from the tolerance model versus both ratios E_2/E_1 and ρ_2/ρ_1, made for material properties distribution functions $\widetilde{\eta}(x)$ described by (6.14)-(6.16) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$.

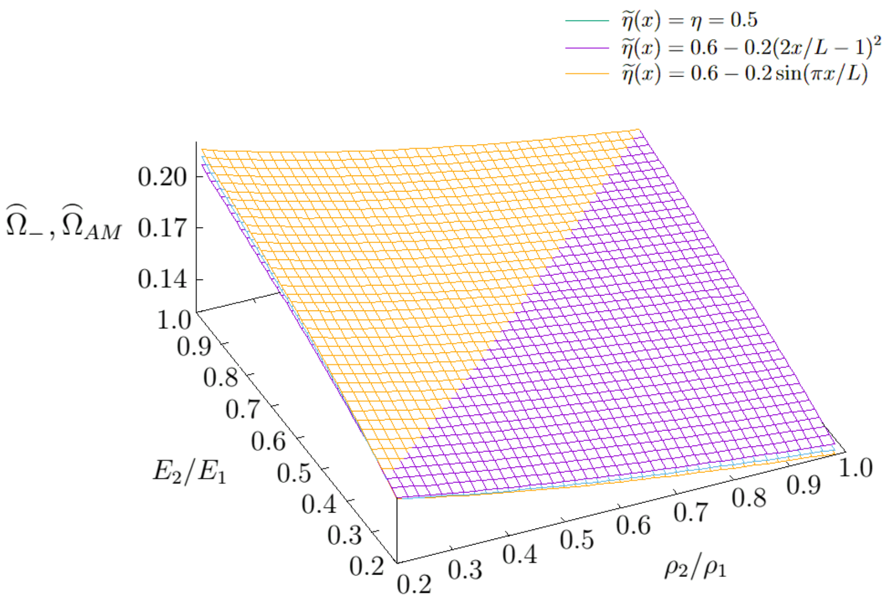

Figure 6.23: Diagrams of dimensionless lower free vibration frequencies $\widehat{\Omega}_-$ (6.56), $\widehat{\Omega}^{AM}$ (6.58) of the shell under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.14)-(6.16) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$

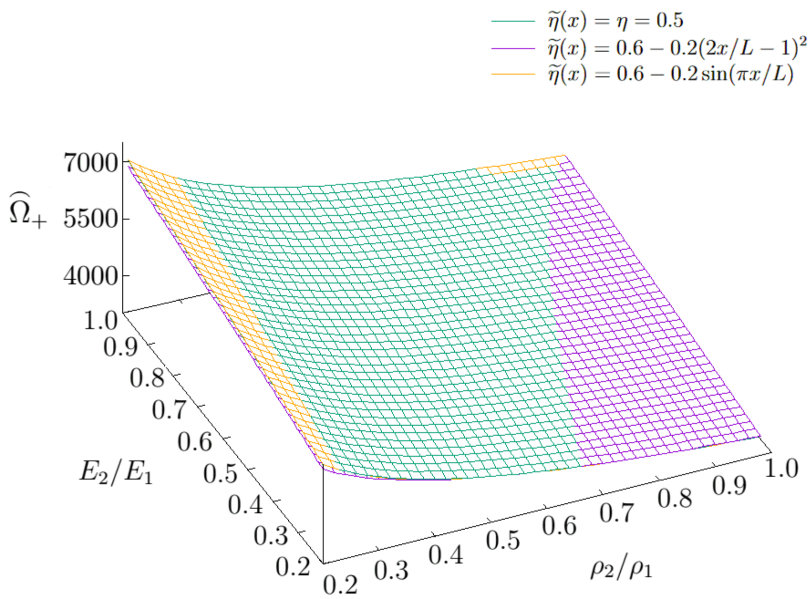

Figure 6.24: Diagrams of dimensionless higher free vibration frequency $\widehat{\Omega}_+$ (6.57) of the shell under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.14)-(6.16) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$

6.3.5 Discussion of analytical and numerical results

On the basis of analytical results obtained in *6.3.2* and *6.3.3* the following important conclusions can be formulated:

- *In the framework of the tolerance model, not only the fundamental lower $\widehat{\omega}_{mn-}$, but also the new additional higher $\widehat{\omega}_{mn+}$ free vibration frequencies can be derived and analysed*; cf. (6.52), (6.53). The higher free vibration frequency is caused by a tolerance-periodic microstructure of the shell under consideration and hence it depends on a microstructure length parameter λ. This frequency cannot be determined using the asymptotic model. Within asymptotic model only the lower cell-independent free vibration frequency $\widehat{\omega}_{mn}^{AM}$ (6.55) can be obtained and investigated.

Similar results have been obtained for the transversally graded shell strip analysed in Subsection 6.2.

On the basis of numerical results shown in Figs. 6.23 and 6.24, the following conclusions can be formulated:

- Values of dimensionless lower free vibration frequencies $\widehat{\Omega}_{-}$ (6.56) and $\widehat{\Omega}^{AM}$ (6.58) calculated in the framework of the tolerance and the asymptotic models are nearly identical, cf. Fig. 6.23. It means, that differences between the values of lower free vibration frequency $\widehat{\Omega}_{-}$ derived from the tolerance model and free vibration frequency $\widehat{\Omega}^{AM}$ obtained from the asymptotic one are negligibly small. Thus, in the problem under consideration, *the effect of microstructure length parameter λ on the "classical" free vibration frequencies can be neglected.* It means that *the asymptotic model governed by equations* (6.54) *is sufficient to determine and investigate free vibration frequencies of the micro-heterogeneous cylindrical shell under consideration.*

- Values of free vibration frequencies Ω_{-}, Ω_{+}, Ω^{AM} increase with the increasing of ratio $E_2/E_1 \in [0.2, 1]$, i.e. with the decreasing of differences between *elastic properties* of the shell component materials, cf. Figs. 6.23 and 6.24. Because the value of Young's module E_1 for the stronger material is fixed then these differences decrease if values of E_2 tend to value of E_1.

- Values of free vibration frequencies Ω_{-}, Ω_{+}, Ω^{AM} decrease with the increasing of ratio $\rho_2/\rho_1 \in [0.2, 1]$, i.e. with the decreasing of differences between *inertial properties* of the component materials, cf. Figs. 6.23 and 6.24. Because the value of mass density ρ_1 for the stronger material is fixed then these differences decrease if values of ρ_2 tend to value of ρ_1.

- The highest values of frequencies $\widehat{\Omega}_-$, $\widehat{\Omega}^{AM}$, cf. Fig. 6.23, are obtained for $\widetilde{\eta}(x)$ given by (6.15), i.e. $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 1.0$, $\rho_2/\rho_1 = 0.2$), i.e. for a shell with a very strong inertial heterogeneity and with elastic homogeneous structure. The smallest values of $\widehat{\Omega}_-$, $\widehat{\Omega}^{AM}$ are obtained for $\widetilde{\eta}(x)$ given by (6.15), i.e. $\widetilde{\eta}(x) = 0.6-0.2\sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 0.2$, $\rho_2/\rho_1 = 1.0$), i.e. for a shell with a very strong elastic heterogeneity and with inertial homogeneous structure.

- The highest value of frequency Ω_+, cf. Fig. 6.24, is obtained for $\widetilde{\eta}(x)$ given by (6.15), i.e. $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 1.0$, $\rho_2/\rho_1 = 0.2$). The smallest value of Ω_+ is obtained for $\widetilde{\eta}(x)$ given by (6.15), i.e. $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 0.2$, $\rho_2/\rho_1 = 1.0$), cf. Fig. 6.24.

6.3.6 Verification of selected analytical results using commercial computer software Ansys

In this subsection the computational results obtained in the framework of the tolerance and asymptotic models will be compared with corresponding results obtained from the commercial software Ansys® Academic Research Mechanical 2020 R1.

Object under consideration is a simply supported thin cylindrical open shell with $r = 1$ m, $L_1 = \pi r/2$, $L_2 = r$, $d = 0.002$ m, $\lambda = L_1/24$. We recall that the shell is made of 2 kinds of materials tolerance-periodically distributed in circurumferential direction as shown in Fig. 6.1. The basic cell is shown in Fig. 6.2. It is assumed that properties of one of these materials are fixed and equal to structural steel properties: $E_1 = 2 \cdot 10^{11}$ Pa, $\rho_1 = 7850$ kg/m^3 $\nu = 0.3$. The properties of the second material are described by $E_2 = E_1\kappa$, $\rho_2 = \rho_1\phi$, $\nu = 0.3$ where $\kappa \in [0.2, 1]$, $\phi \in [0.2, 1]$. Calculations are made for two distribution functions of material properties: $\widetilde{\eta}(x) = 0.6-0.2(2x/L-1)^2$ and $\widetilde{\eta}\,(x) = 0.6-0.2\sin(\pi x/L)$. We recall that $L \equiv L_1$.

In Fig. 6.25 there are presented diagrams of lower free vibration frequency $\widehat{\omega}_-$ (6.56) derived from the tolerance model versus both ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ described by (6.14), (6.15) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$. Note, that diagrams shown in Fig. 6.25 are the same as shown in Fig. 6.23, but made for dimension values of free vibration frequency $\widehat{\omega}_-$. Equation (6.59) given below shows how these values were calculated (units conversion included)

$$
\begin{aligned}
\widehat{\omega}_{-} &= \frac{1}{2\pi}\sqrt{\frac{E_1}{(1-\nu^2)\,\rho_1 L_1^2}\left(\widehat{\Omega}_{-}\right)^2} = \\
&= \frac{1}{2\pi}\sqrt{\frac{2\cdot 10^{11}\mathrm{Pa}}{\left(1-(0.3)^2\right)7850\ \mathrm{kg/m^3}\ (1.57\ \mathrm{m})^2}\left(\widehat{\Omega}_{-}\right)^2} = \\
&= 536.12\ \frac{1}{s}\ \widehat{\Omega}_{-}
\end{aligned}
\tag{6.59}
$$

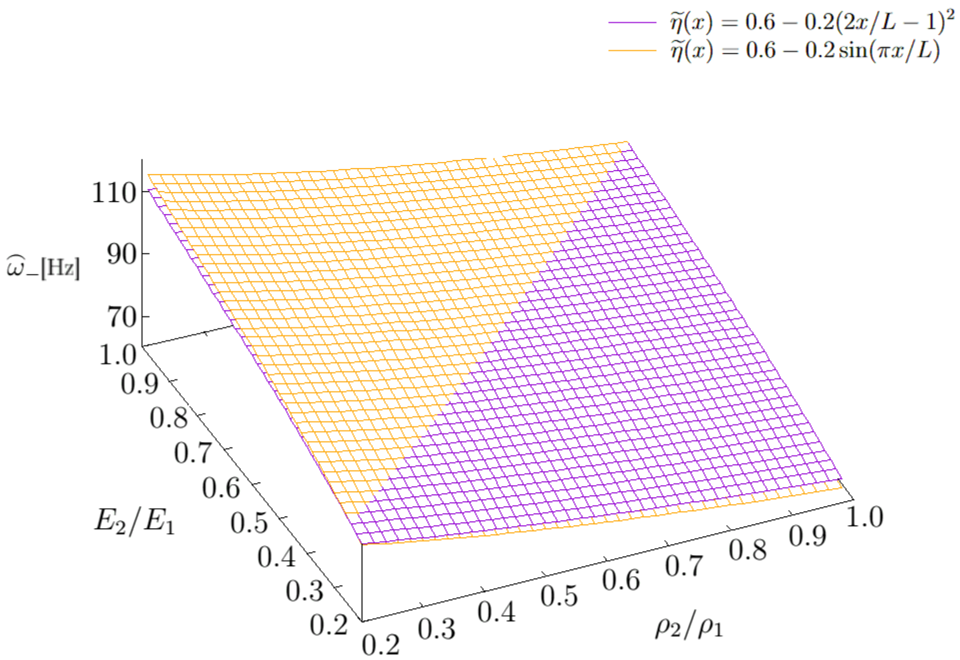

Figure 6.25: Diagrams of lower free vibration frequency $\widehat{\omega}_{-}$ (6.59) of the shell under consideration versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.14), (6.15) and for $\lambda/L_1 = 1/24$, $d/\lambda = 12/(125\pi)$

Using Ansys, the eight node quadrilateral shaped shell element (SHELL281) was applied for meshing the shell. Boundary conditions were modelled by fixed displacements in the respective directions. Mutual contact regions were used to model perfect boundary on interfaces.

For every considered pair E_2/E_1, ρ_2/ρ_1 and for both distribution functions the first (smallest) fundamental free vibration frequency is obtained for wave numbers $\alpha = 5\pi/L_1$ and $\beta = \pi/L_2$ (i.e. for $m = 5$ and $n = 1$). The first (i.e. lowest) free vibration modes obtained for $E_2/E_1 = 0.5$ and $\rho_2/\rho_1 = 0.5$, for $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ and $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$ are presented in Figs. 6.27 and 6.28, respectively.

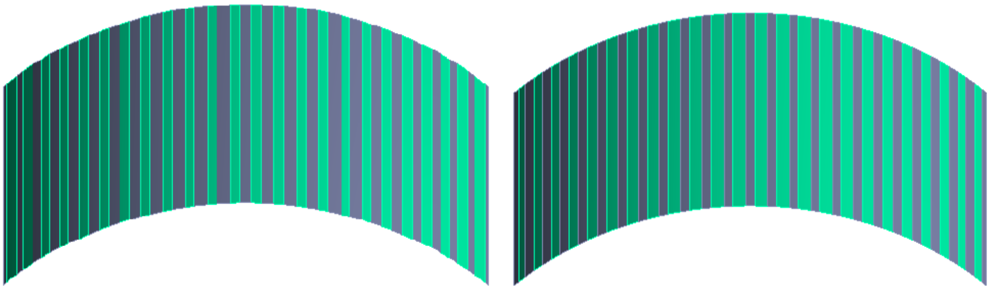

Figure 6.26: Shells under consideration shown in commercial software Ansys; a) tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ b) tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$

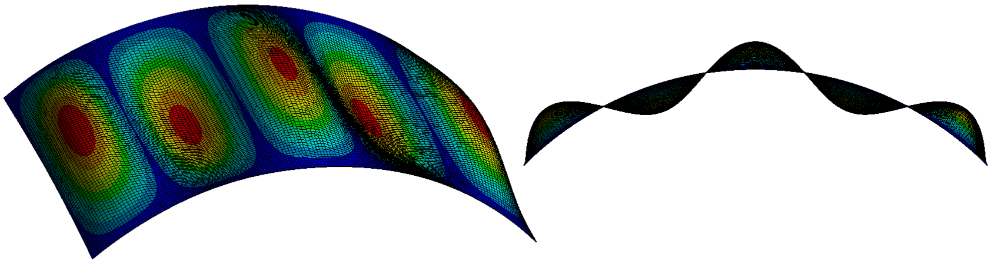

Figure 6.27: The first free vibration mode related to distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$

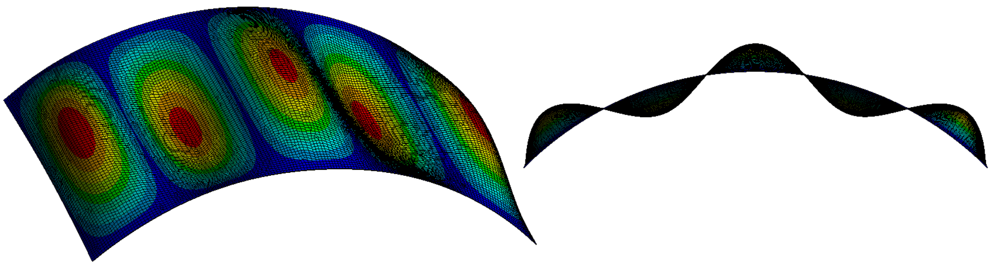

Figure 6.28: The first free vibration mode related to distribution function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$

Results shown in Figs. 6.27 and 6.28 proof that the lowest vibration mode related to distributions functions given by (6.14) and (6.15) are nearly identical.

The first step was a convergence analysis for the results obtained from Ansys, i.e. to check weather the increase in the number of elements has a significant impact on the values of vibration frequencies. Calculations were made for $E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$, for both distrubution functions under consideration and for different number of finite elements. The results are shown in Figs. 6.29 and 6.30. In the next steps calculations are made for 8019 elements for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ and for 8020 elements for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$.

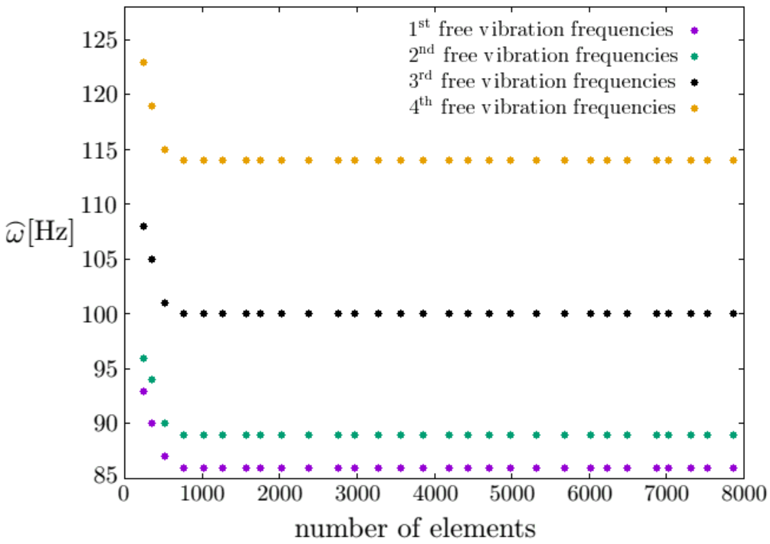

Figure 6.29: Diagrams of the first four free vibration frequencies of the shell under consideration with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6-0.2(2x/L-1)^2$ obtained from commercial software Ansys versus number of elements used for calculations

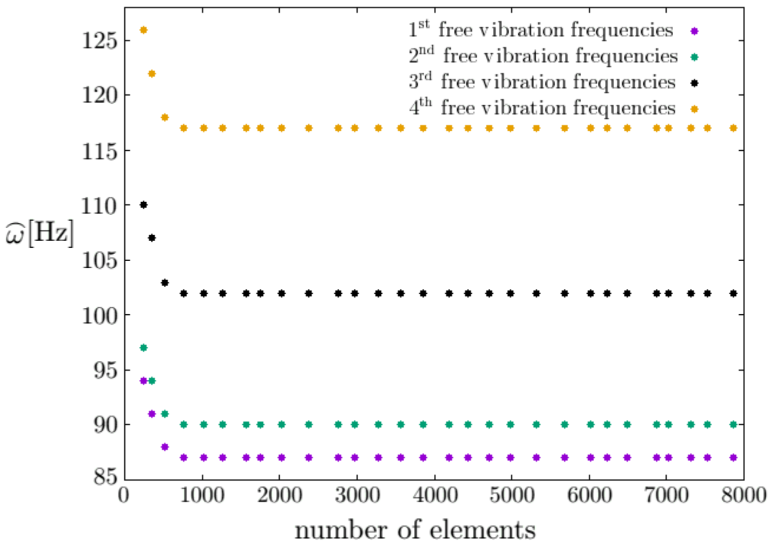

Figure 6.30: Diagrams of the first four free vibration frequencies of the shell under consideration with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6-0.2\sin(\pi x/L)$ obtained from commercial software Ansys versus number of elements used for calculations

It is worth noting that for the shell under consideration the difference between first and second free vibration frequencies is relatively small (approximately 1.5%), so from an engineering point of view it is reasonable to compare the results for a larger number of succesive free vibration frequencies. In the next steps the first 25 smallest free vibration frequencies are considered. We define the following mean absolute relative error (MARE)

$$MARE = \frac{1}{25}\sum_{n=1}^{25}\left|\frac{\widehat{\omega}_i^{\mathrm{FEM}} - \widehat{\omega}_i^{TM}}{\widehat{\omega}_i^{\mathrm{FEM}}}\right| \tag{6.60}$$

where $\widehat{\omega}_i^{\mathrm{FEM}}$ is i-th free vibration frequency obtained from Finite Element Method, $\widehat{\omega}_i^{TM}$ is i-th free vibration frequency obtained from Tolerance Modelling.

Below we present computational results for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L_1 - 1)^2$.

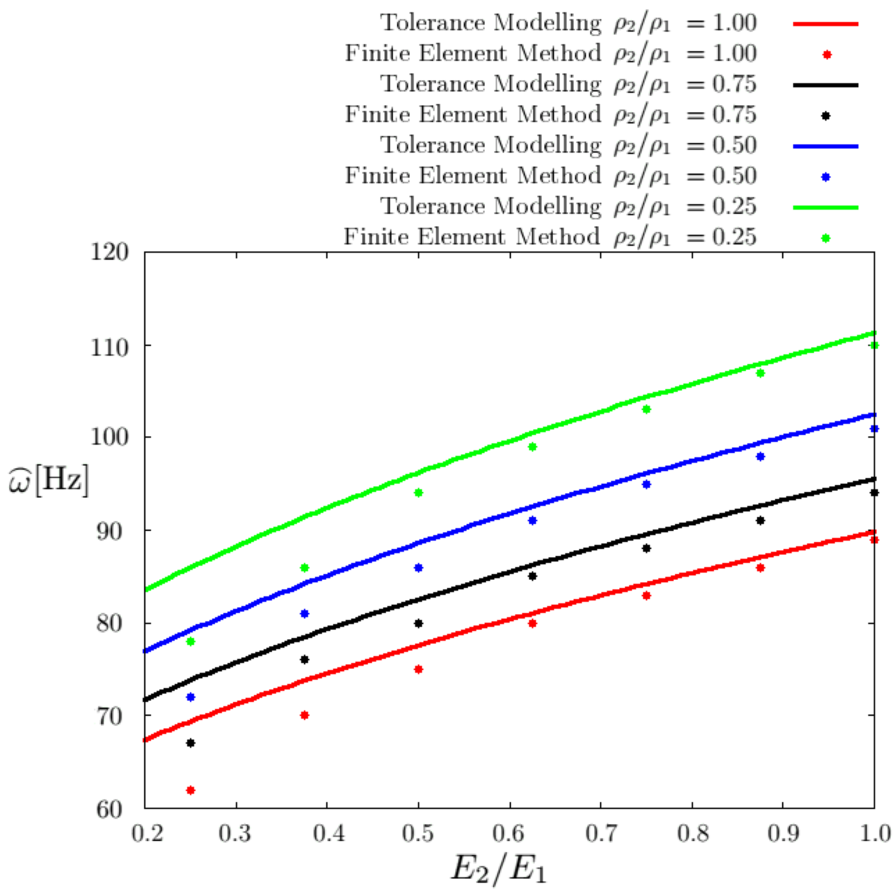

Figure 6.31: Comparision of results obtained from the tolerance modelling and from the finite element method (Ansys), for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ and fixed values $\rho_2/\rho_1 = \{0.25, 0.5, 0.75, 1.0\}$

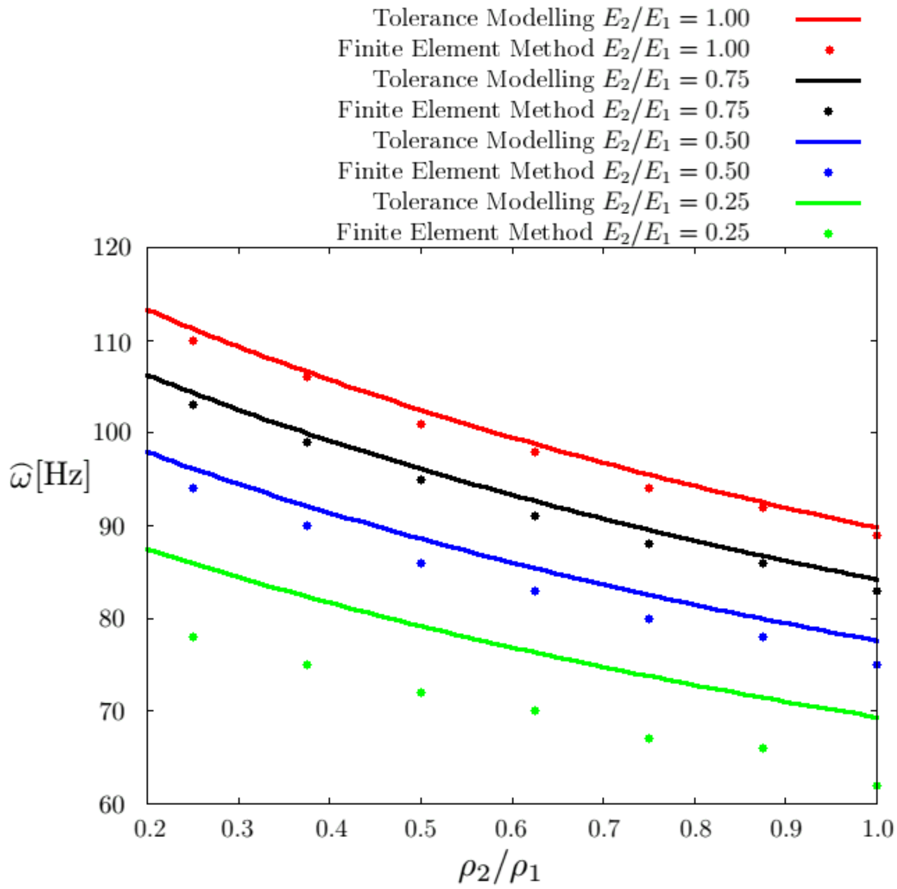

Figure 6.32: Comparision of results obtained from the tolerance modelling and from the finite element method (Ansys), for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ and fixed values $E_2/E_1 = \{0.25, 0.5, 0.75, 1.0\}$

		E_2/E_1			
		0.25	0.50	0.75	1.00
ρ_2/ρ_1	0.25	5.19%	1.8%	1.06%	1.04%
	0.50	5.46%	1.80%	0.97%	0.86%
	0.75	5.71%	1.89%	0.97%	0.79%
	1.00	5.94%	1.99%	1.01%	0.77%

Table 6.1: Relative error for first free frequency for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$, for different values of pairs E_2/E_1 and ρ_2/ρ_1

		E_2/E_1			
		0.25	0.50	0.75	1.00
ρ_2/ρ_1	0.25	6.39%	1.78%	0.70%	0.62%
	0.50	7.28%	1.9%	0.60%	0.46%
	0.75	7.58%	2.07%	0.61%	0.41%
	1.00	7.81%	2.21%	0.64%	0.41%

Table 6.2: Mean absolute relative error (6.60) for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$, for different values of pairs E_2/E_1 and ρ_2/ρ_1

Below we present computational results for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$.

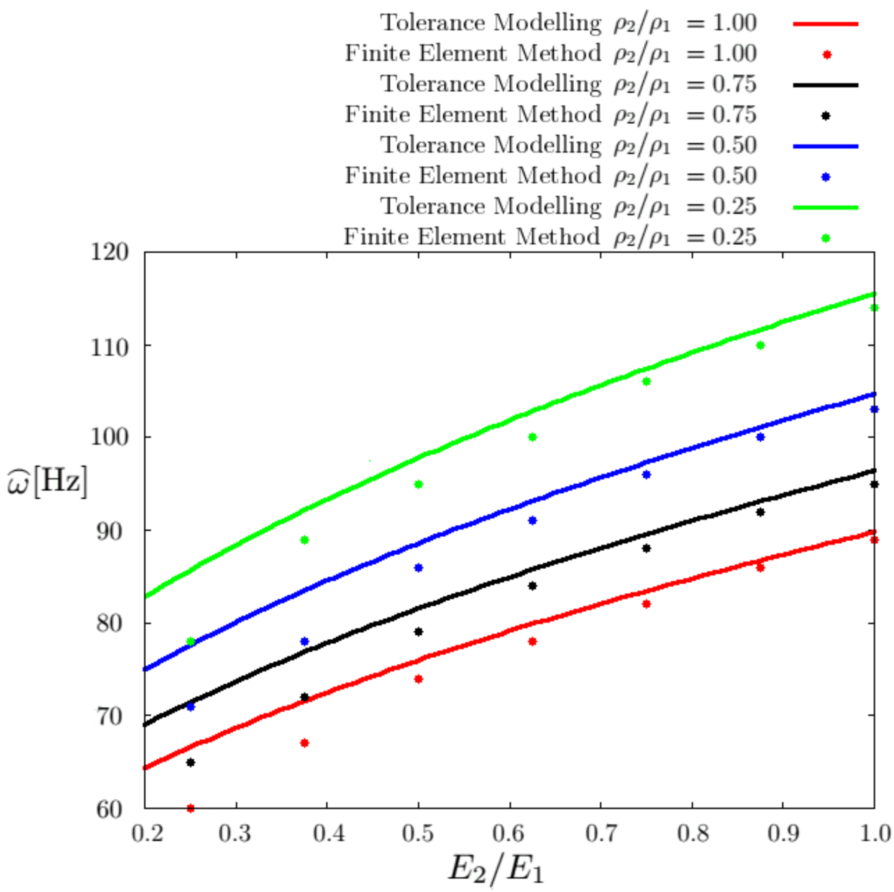

Figure 6.33: Comparision of results obtained from the tolerance modelling and from the finite element method (Ansys), for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and fixed values $\rho_2/\rho_1 = \{0.25, 0.5, 0.75, 1.0\}$

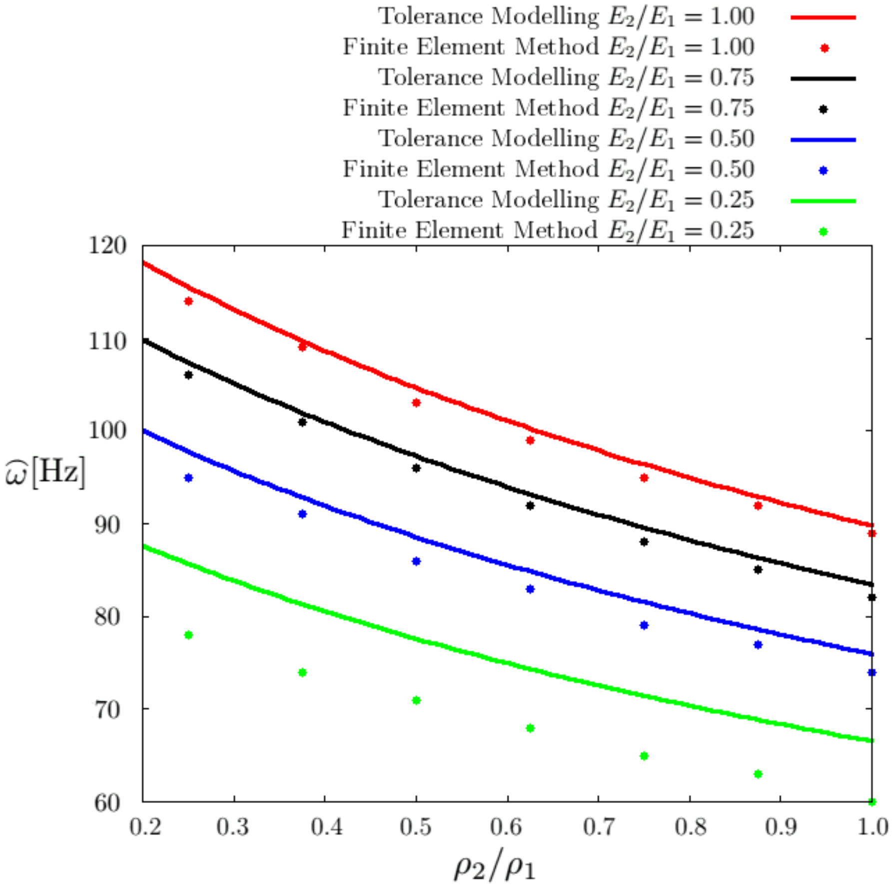

Figure 6.34: Comparision of results obtained from the tolerance modelling and from the finite element method (Ansys), for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$ and fixed values $E_2/E_1 = \{0.25, 0.5, 0.75, 1.0\}$

		E_2/E_1			
		0.25	0.50	0.75	1.00
ρ_2/ρ_1	0.25	4.37%	1.64%	1.12%	1.15%
	0.50	4.65%	1.64%	0.98%	0.90%
	0.75	4.92%	1.74%	0.96%	0.80%
	1.00	5.15%	1.87%	1.00%	0.78%

Table 6.3: Relative error for first free frequency for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6 - 0.2\sin(\pi x/L)$, for different values of pairs E_2/E_1 and ρ_2/ρ_1

		E_2/E_1			
		0.25	0.50	0.75	1.00
ρ_2/ρ_1	0.25	5.98%	1.57%	0.68%	0.62%
	0.50	6.43%	1.76%	0.60%	0.47%
	0.75	6.74%	1.93%	0.60%	0.41%
	1.00	6.96%	2.06%	0.62%	0.41%

Table 6.4: Mean absolute relative error (6.60) for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$, for different values of pairs E_2/E_1 and ρ_2/ρ_1

Discussion of results

On the basis of results shown in this subsection the following conclusions can be formulated:

- The high convergence of results (cf. Figs. 6.30-6.29) makes it possible to use results obtained from Ansys as a reference to those obtained in the framework of the tolerance modelling procedure.

- For both distribution functions under consideration the first fundamental frequency was $\widehat{\omega}_{51}$. The difference between first and second free vibration frequencies is relatively small, so it is important not to limit calculations to only first fundamental vibration frequency.

- The values of ratio E_1/E_2 have a greater impact on values of both the relative error for first free frequency and the mean absolute relative error than the values of ratio ρ_2/ρ_1.

- Relative error for the first free frequency for the tolerance-periodic shell with distribution of material properties given by function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ varies from 0.77% (for $E_2/E_1 = 1.0$ and $\rho_2/\rho_1 = 1.0$) to 5.94% (for $E_2/E_1 = 0.25$ and $\rho_2/\rho_1 = 1.0$). Mean absolute relative error for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$ varies from 0.77% (for $E_2/E_1 = 1.0$ and $\rho_2/\rho_1 = 1.0$) to 5.15% (for $E_2/E_1 = 0.25$ and $\rho_2/\rho_1 = 1.0$).

- For shells with small differences in material properties $E_1/E_2 \in [0.5, 1]$ and $\rho_2/\rho_1 \in [0.5, 1]$, maximum relative error for the first free frequency for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ is equal to 1.99% and for distribution function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$ is equal to 1.87%.

- Mean absolute relative error for the tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ varies from 0.41% (for $E_2/E_1 = 1.0$ and $\rho_2/\rho_1 = 1.0$) to 7.81% (for $E_2/E_1 = 0.25$ and $\rho_2/\rho_1 = 1.0$). Mean absolute relative error for tolerance-periodic shell with distribution of material properties described by function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$ varies from 0.41% (for $E_2/E_1 = 1.0$ and $\rho_2/\rho_1 = 1.0$) to 6.96% (for $E_2/E_1 = 0.25$ and $\rho_2/\rho_1 = 1.0$).

- For shells with small differences in material properties $E_1/E_2 \in [0.5, 1]$ and $\rho_2/\rho_1 \in [0.5, 1]$, maximum mean absolute relative error for distribution function $\widetilde{\eta}(x) = 0.6 - 0.2(2x/L - 1)^2$ is equal to 2.21% and for distribution function $\widetilde{\eta}\,(x) = 0.6 - 0.2\sin(\pi x/L)$ is equal to 2.06%.

7. Selected dynamic problems of micro-dynamics: Application of the combined asymptotic-tolerance model

7.1. Introduction

In all special micro-dynamic problems investigated in this chapter, the object of considerations is an open thin cylindrical shell with L_1, L_2, r, d as its circumferential length, axial length, midsurface curvature radius and constant thickness, respectively. The shell has a functionally graded macrostructure and a tolerance-periodic microstructure along circumferential direction as well as a constant structure in the axial direction. On the microscopic level, the shell is made of two elastic isotropic materials, which are perfectly bonded on interfaces and tolerance-periodically distributed along x-coordinate. Such a shell is shown in Fig. 6.1.

The basic cell defined by $\Delta \equiv [-\lambda/2, \lambda/2]$, cf. definition (6.1), is shown in Fig. 6.2.

Properties of the component materials are described by Young's moduli E_1, E_2, Poisson's ratio $\nu \equiv \nu_1 = \nu_2$ and mass densities ρ_1, ρ_2. Inside the cell, the elastic $E \in TP^0_\delta(\Omega, \Delta)$ and inertial $\rho \in TP^0_\delta(\Omega, \Delta)$ properties of the shell have periodic approximations $\widetilde{E}(x,z)$, $\widetilde{\rho}(x,z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, defined by (6.2).

The rigidities $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $x \in \Omega$, of the shell are described in Subsection 6.1.

The considerations will be based on *superimposed microscopic model equations* (5.34)-(5.36) derived in the second step of the combined asymptotic-tolerance modelling. Equations (5.34)-(5.36) are independent of solutions obtained in the framework of asymptotic (macroscopic model) derived in the first step of combined modelling. Hence, they make it possible to investigate the shell micro-dynamics separately from the shell macro-dynamics. *This is the greatest advantage of the combined asymptotic-tolerance model proposed in this dissertation.*

The considerations will be restricted to the simplest form of the model in which $a = n = A = N = 1$.

Periodic approximations $\widetilde{h}(x,z)$, $\widetilde{g}(x,z)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, of fluctuation shape functions $h \in FS^1_\delta(\Omega,\Delta)$, $g \in FS^2_\delta(\Omega,\Delta)$ are given by (6.3) and (6.4), respectively.

Some from approximations $\widetilde{\eta}(x)$ of material properties distribution functions $\eta(x)$ expressed by (6.6)-(6.16) will be used.

In this subsection, the influence of a cell size on the free micro-vibration frequencies and on the character of displacement micro-fluctuations caused by a tolerance-periodic structure of the shell will be investigated. Moreover, the effect of a microstructure size on the displacement wave propagating in the axial direction, i.e. in the direction parallel to the interfaces between component materials, will be studied. It will be also shown that *the superimposed microscopic model equations* (5.34)-(5.36) describe certain *space-boundary layer phenomena* strictly related to the specific form of boundary conditions imposed on micro-fluctuation amplitudes being unknowns in these equations. The length-scale effect will be also analysed in a certain special initial value problem.

It has to be emphasized that the special problems mentioned above cannot be analysed in the framework of asymptotic models.

7.2. Free micro-vibrations

In this subsection we derive micro-vibration frequencies of the tolerance-periodic shell under consideration independently of the macro-vibration frequencies. The shell is simply supported on all four edges.

The subsequent analysis will be based on Eqs. (5.34)-(5.36). Since coefficients of these equations are functions of x, then the approximate formulae of free vibration frequencies will be derived applying the known Galerkin method, cf. [46], in the range $0 \leq x \leq L_1$.

7.2.1 Analytical results

Free micro-vibrations in circumferential direction

The shell free micro-vibrations along circumferential direction are described by Eq. (5.34). For $a = n = 1$, this equation has the following form

$$\underline{\langle D^{1212}(h)^2\rangle}(x)\partial_{22}Q_1 - \left\langle D^{1111}\left(\partial_1 h\right)^2\right\rangle(x)Q_1 - \underline{\left\langle \mu\left(h\right)^2\right\rangle}(x)\ddot{Q}_1 = 0, \qquad x \in \Omega_\Delta, \tag{7.1}$$

where averages $\langle D^{1111}(\partial_1 h)^2\rangle$, $\langle D^{1212}(h)^2\rangle$, $\langle \mu(h)^2\rangle$ are given in Appendix, cf. expressions (A.4), (A.5) and (A.20). The underlined terms in (7.1) depend on a cell size λ.

Solution to Eq. (7.1) satisfying the boundary conditions for a shell simply supported on edges $\xi = 0$, $\xi = L_2$ can be assumed in the form

$$Q_1(x, \xi, t) = Q_1^*(x) \sin\left(\pi\xi / L_2\right) \cos(\overline{\omega} t), \tag{7.2}$$

where $Q_1^*(x)$ is a slowly-varying function in x satisfying boundary conditions on edges $x = 0$, $x = L_1$ and $\overline{\omega}$ is a frequency of free micro-vibrations along circumferential direction. Substituting (7.2) into (7.1), for $\sin(\pi\xi/L_2) \neq 0$, we arrive at equation for $Q_1^*(x)$

$$\begin{gathered}Q_1^*(x)\Big[-\left(\pi/L_2\right)^2 \langle D^{1212}(h)^2\rangle(x) - \left\langle D^{1111}\left(\partial_1 h\right)^2\right\rangle(x) + \\ +\overline{\omega}^2 \left\langle \mu\,(h)^2 \right\rangle(x)\Big] = 0,\end{gathered} \tag{7.3}$$

In order to obtain approximate formula of free vibration frequency $\overline{\omega}$, the known Galerkin method, cf. [46], can be applied to Eq. (7.3). Solution to Eq. (7.3) satisfying the boundary conditions for a shell simply supported on edges $x = 0$, $x = L_1$ is assumed as $Q_1^*(x) = A\cos(\pi x/L_1)$. We substitute this solution into (7.3). For $A \neq 0$, the orthogonality condition of the resulting left-hand side of Eq. (7.3) and function $\cos(\pi x/L_1)$ has the following form

$$\begin{gathered}\int_0^{L_1}\Big[-\left(\pi/L_2\right)^2 \langle D^{1212}(h)^2\rangle(x) - \left\langle D^{1111}\left(\partial_1 h\right)^2\right\rangle(x) + \\ +\overline{\omega}^2 \left\langle \mu\,(h)^2 \right\rangle(x)\Big]\cos^2\left(\pi x/L_1\right)dx = 0,\end{gathered}$$

Setting $\overline{h} = \lambda^{-1}h$, from the above orthogonality condition, we obtain the following formula for $\overline{\omega}$

$$\begin{gathered}\overline{\omega}^2 = \left[\lambda^2\int_0^{L_1}\left\langle\mu(\overline{h})^2\right\rangle(x)\cos^2\left(\pi x/L_1\right)dx\right]^{-1}\int_0^{L_1}\Big[-\left(\pi/L_2\right)^2\lambda^2\left\langle D^{1212}(\overline{h})^2\right\rangle(x) + \\ -\left\langle D^{1111}\left(\partial_1 h\right)^2\right\rangle(x)\Big]\cos^2\left(\pi x/L_1\right)dx.\end{gathered} \tag{7.4}$$

Free micro-vibrations in axial direction

The shell free micro-vibrations in axial direction are described by Eq. (5.35). For $a = n = 1$, this equation has the following form

$$\underline{\langle D^{2222}(h)^2\rangle}(x)\partial_{22}Q_2 - \left\langle D^{1212}\left(\partial_1 h\right)^2\right\rangle(x)Q_2 - \underline{\left\langle \mu\left(h\right)^2\right\rangle}(x)\ddot{Q}_2 = 0, \qquad x \in \Omega_\Delta, \tag{7.5}$$

where averages $\langle D^{1212}(\partial_1 h)^2\rangle$, $\langle D^{2222}(h)^2\rangle$, $\langle\mu(h)^2\rangle$ are given in Appendix, cf. expressions (A.6), (A.7) and (A.20). The underlined terms in (7.5) depend on a microstructure length parameter λ.

Solution to Eq. (7.5) satisfying the boundary conditions for a shell simply supported on edges $\xi = 0$, $\xi = L_2$ can be assumed in the form

$$Q_2\left(x, \xi, t\right) = Q_2^*\left(x\right)\cos\left(\pi\xi/L_2\right)\cos\left(\breve{\omega}t\right), \tag{7.6}$$

where $Q_2^*(x)$ is a slowly-varying function in x satisfying boundary conditions on edges $x = 0$, $x = L_1$ and $\breve{\omega}$ is a frequency of free micro-vibrations along axial direction. Substituting (7.6) into (7.5), for $\cos(\pi\xi/L_2) \neq 0$, we arrive at equation for $Q_2^*(x)$

$$Q_2^*\left(x\right)\left[-\left(\pi/L_2\right)^2\langle D^{2222}(h)^2\rangle\left(x\right) - \left\langle D^{1212}\left(\partial_1 h\right)^2\right\rangle\left(x\right) + \breve{\omega}^2\left\langle \mu\left(h\right)^2\right\rangle\left(x\right)\right] = 0, \tag{7.7}$$

Solution to Eq. (7.7) satisfying the boundary conditions for a shell simply supported on edges $x = 0$, $x = L_1$ is assumed as $Q_2^* = B\sin(\pi x/L_1)$. We substitute this solution into (7.7). By means of Galerkin method, for $B \neq 0$, the following orthogonality condition of the resulting left-hand side of Eq. (7.7) and function $\sin(\pi x/L_1)$ is obtained

$$\int_0^{L_1}\left[-\left(\pi/L_2\right)^2\langle D^{2222}(h)^2\rangle\left(x\right) - \left\langle D^{1212}\left(\partial_1 h\right)^2\right\rangle\left(x\right) + \breve{\omega}^2\left\langle \mu\left(h\right)^2\right\rangle\left(x\right)\right]\sin^2\left(\pi x/L_1\right)dx = 0,$$

Setting $\overline{h} = \lambda^{-1}h$, from the above orthogonality condition, we derive the following formula for $\breve{\omega}$

$$\breve{\omega}^2 = \frac{\int_0^{L_1}\left[-\left(\pi/L_2\right)^2\lambda^2\left\langle D^{2222}(\overline{h})^2\right\rangle\left(x\right) - \left\langle D^{1212}\left(\partial_1 h\right)^2\right\rangle\left(x\right)\right]\sin^2\left(\pi x/L_1\right)dx}{\lambda^2\int_0^{L_1}\left\langle \mu(\overline{h})^2\right\rangle\left(x\right)\sin^2\left(\pi x/L_1\right)dx}. \tag{7.8}$$

Transversal free micro-vibrations

Free transversal micro-vibrations of the shell under consideration are described by Eq. (5.36). For $A = N = 1$, this equation has the following form

$$\underline{\langle B^{2222}(g)^2\rangle}(x)\partial_{2222}V + \left[2\underline{\langle B^{1122}g\partial_{11}g\rangle}(x) - 4\underline{\left\langle B^{1212}\left(\partial_1 g\right)^2\right\rangle}(x)\right]\partial_{22}V + \\ + \left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle(x)V + \underline{\left\langle \mu\left(g\right)^2\right\rangle}(x)\ddot{V} = 0, \tag{7.9}$$

where averages occurring in the above equations are given in Appendix, cf. (A.13), (A.16)-(A.18) and (A.21). The underlined terms in (7.9) depend on a cell size λ.

Solution to Eq. (7.9) satisfying the boundary conditions for a shell simply supported on edges $\xi = 0$, $\xi = L_2$ can be assumed in the form

$$V\left(x,\xi,t\right) = V^*\left(x\right)\sin\left(\pi\xi/L_2\right)\cos\left(\omega t\right), \tag{7.10}$$

where $V^*(x)$ is a slowly-varying function in x satisfying boundary conditions on edges $x = 0$, $x = L_1$ and ω is a frequency of free transversal micro-vibrations. Substituting (7.10) into (7.9), for $\sin(\pi\xi/L_2) \neq 0$, we arrive at equation for $V^*(x)$

$$V^*\left(x\right)\Bigg[\left(\pi/L_2\right)^4\langle B^{2222}(g)^2\rangle\left(x\right) - 2\left(\pi/L_2\right)^2\Big(\langle B^{1122}g\partial_{11}g\rangle\left(x\right) + \\ -2\left\langle B^{1212}\left(\partial_1 g\right)^2\right\rangle\left(x\right)\Big) + \left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle\left(x\right) - \omega^2\left\langle \mu\left(g\right)^2\right\rangle\left(x\right)\Bigg] = 0, \tag{7.11}$$

Solution to Eq. (7.11) satisfying the boundary conditions for a shell simply supported on edges $x = 0$, $x = L_1$ is assumed as $V^*(x) = C\sin(\pi x/L_1)$. We substitute this solution into (7.11). By means of Galerkin method, for $C \neq 0$ the orthogonality condition of the resulting left-hand side of Eq. (7.11) and function $\sin(\pi x/L_1)$ is obtained. Setting $\widehat{\mathrm{g}}(\cdot) = \lambda^{-1}g(\cdot)$, $\overline{g}(\cdot) = \lambda^{-2}g(\cdot)$, from this orthogonality condition, we derive the following formula for ω

$$\omega^2 = \left[\lambda^4\int\limits_0^{L_1}\left\langle \mu\left(\overline{g}\right)^2\right\rangle\left(x\right)\sin^2\left(\pi x/L_1\right)dx\right]^{-1}\int\limits_0^{L_1}\Bigg[\left(\pi/L_2\right)^4\lambda^4\langle B^{2222}(\overline{g})^2\rangle\left(x\right) + \\ - 2\left(\pi/L_2\right)^2\lambda^2\left(\langle B^{1122}\overline{g}\partial_{11}g\rangle\left(x\right) - 2\left\langle B^{1212}\left(\partial_1\widehat{\mathrm{g}}\right)^2\right\rangle\left(x\right)\right) + \\ + \left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle\left(x\right)\Bigg]\sin^2\left(\pi x/L_1\right)dx. \tag{7.12}$$

7.2.2 Numerical calculations

All calculations are made using Maple by Maplesoft software and all charts are made in gnuplot.

For the shell under consideration, calculations are made for approximations $\widetilde{\eta}(x)$ of material properties distribution functions $\eta(x)$ given by (6.6), (6.7), (6.10), (6.16), i.e. for $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \sin(\pi x/L)$, $\widetilde{\eta}(x) = \eta = 0.5$. We recall that $L \equiv L_1$

Diagrams of these functions are shown in Fig. 6.3. Additionally, distribution of material properties described by functions applied in the problem analysed here are shown in Fig. 7.1.

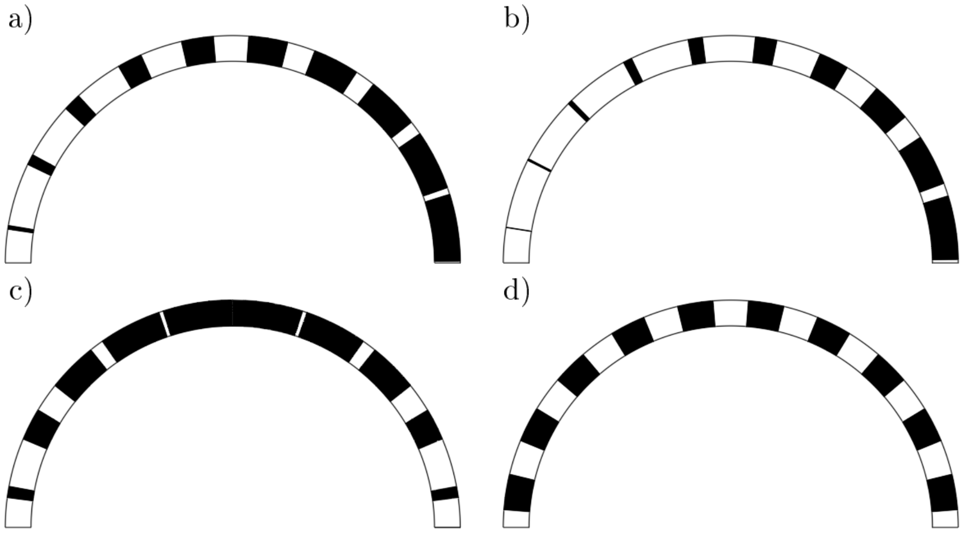

Figure 7.1: Distribution of materials described by a) $\widetilde{\eta}(x) = x/L$ b) $\widetilde{\eta}(x) = (x/L)^2$ c) $\widetilde{\eta}(x) = \sin(\pi x/L)$ d) $\widetilde{\eta}(x) = \eta = 0.5$, we recall that $L \equiv L_1$

In the subsequent analysis, denotation $L \equiv L_1$ will be used.

We define the following dimensionless free micro-vibration frequencies

$$\left(\overline{\Omega}\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1} \left(\overline{\omega}\right)^2 , \tag{7.13}$$

$$\left(\breve{\Omega}\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1} \left(\breve{\omega}\right)^2 , \tag{7.14}$$

$$\left(\Omega\right)^2 \equiv \frac{\left(1-\nu^2\right)\rho_1 L^2}{E_1} \left(\omega\right)^2 , \tag{7.15}$$

where frequencies $\overline{\omega}$, $\breve{\omega}$, ω are determined by formulae (7.4), (7.8) and (7.12), respectively.

Some numerical results calculated by formulae (7.13)-(7.15) are shown in Figs. 7.2-7.8.

Calculations are made for Poisson's ratio $\nu = 0.3$, for fixed ratios $L_2/L = 2$, $d/\lambda = 0.1$ and for various ratios $\lambda/L \in [0.01, 0.1]$, $E_2/E_1 \in [0.2, 1.0]$, $\rho_2/\rho_1 \in [0.2, 1.0]$.

From expressions (7.13)-(7.15) it follows that all plots are made under assumption $L \equiv L_1 = \text{const}$, $E_1 = \text{const}$, $\rho_1 = \text{const}$.

In Figs. 7.2-7.4 there are presented diagrams of dimensionless free micro-vibration frequencies $\overline{\Omega}$, $\breve{\Omega}$, Ω given by (7.13)-(7.15), respectively, versus both ratios E_2/E_1 and ρ_2/ρ_1, made for $\lambda/L = 0.01$, $d/\lambda = 0.1$ and for distribution functions of material properties $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \sin(\pi x/L)$, $\widetilde{\eta}(x) = \eta = 0.5$. Note that diagram for distribution function $\widetilde{\eta}(x) = x/L$ is not visible because it is very similar to diagram for distribution function $\widetilde{\eta}(x) = \eta = 0.5$.

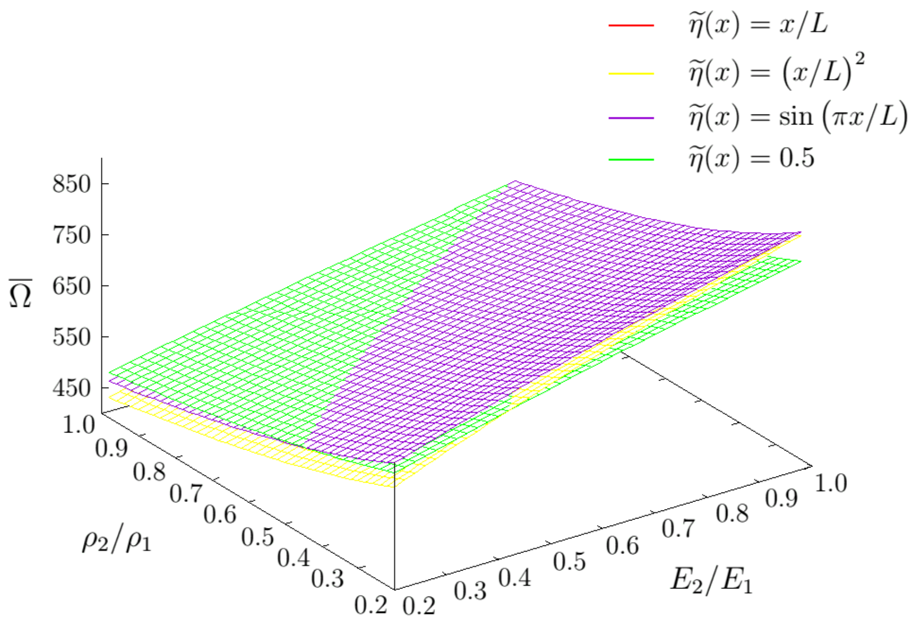

Figure 7.2: Diagrams of dimensionless free micro-vibration frequency $\overline{\Omega}$ (7.13) versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L = 0.01$

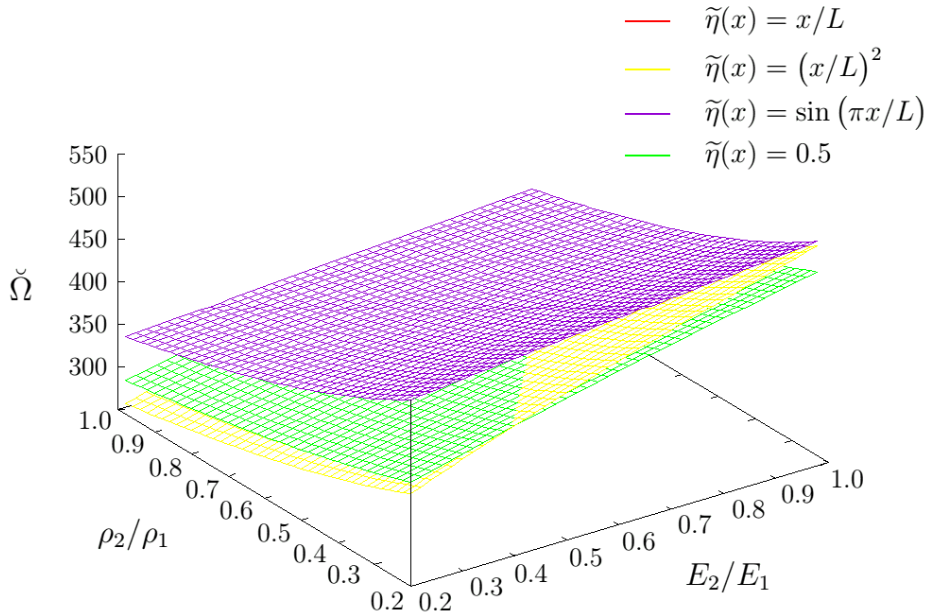

Figure 7.3: Diagrams of dimensionless free micro-vibration frequency $\breve{\Omega}$ (7.14) versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L = 0.01$

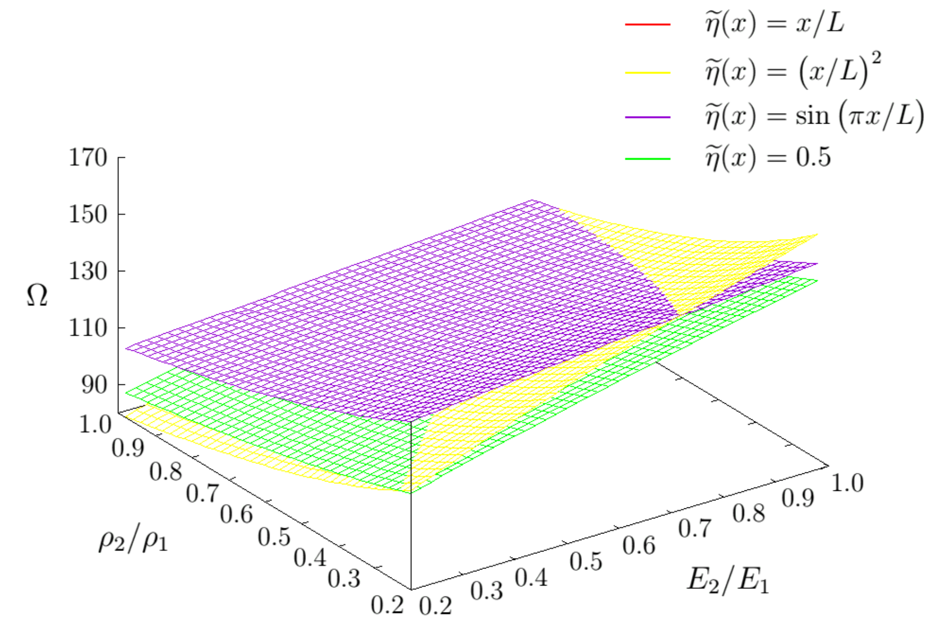

Figure 7.4: Diagrams of dimensionless free micro-vibration frequency Ω (7.15) versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L = 0.01$

In Figs. 7.5-7.7 there are shown plots of dimensionless free micro-vibration frequencies $\overline{\Omega}$, $\breve{\Omega}$, Ω given by (7.13)-(7.15), respectively, versus dimensionless microstructure length parameter λ/L, made for $E_2/E_1 = 0.5$, $\rho_2/\rho_1 = 0.5$, $d/\lambda = 0.1$ and for distribution functions of material properties $\widetilde{\eta}(x)$ given by $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \sin(\pi x/L)$, $\widetilde{\eta}(x) = \eta = 0.5$. Note that diagram for distribution function $\widetilde{\eta}(x) = x/L$ is not visible because it is very similar to diagram for distribution function $\widetilde{\eta}(x) = \eta = 0.5$.

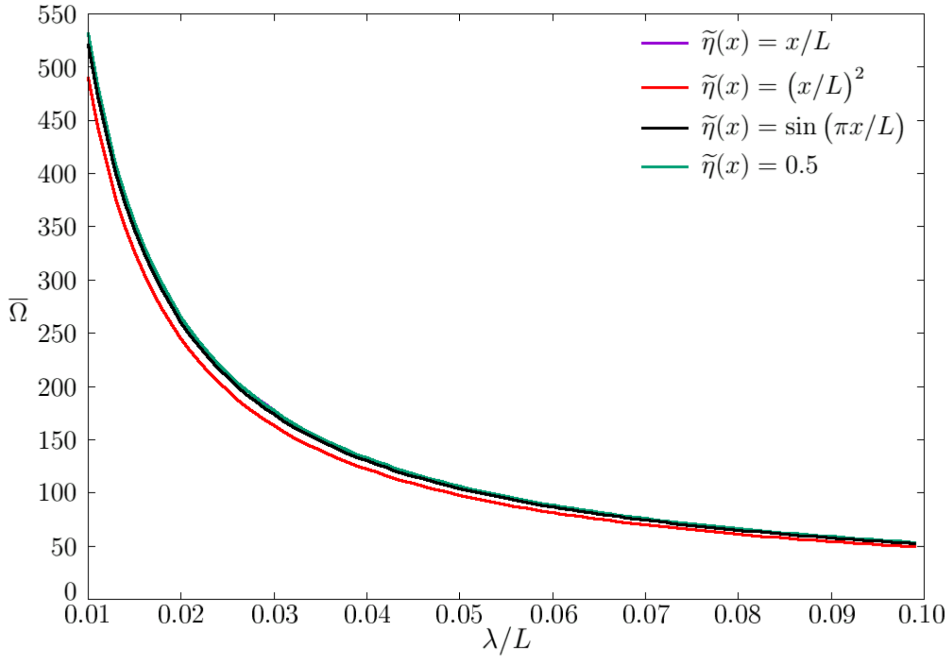

Figure 7.5: Diagrams of dimensionless free micro-vibration frequency $\overline{\Omega}$ (7.13) versus dimensionless microstructure length parameter λ/L, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$

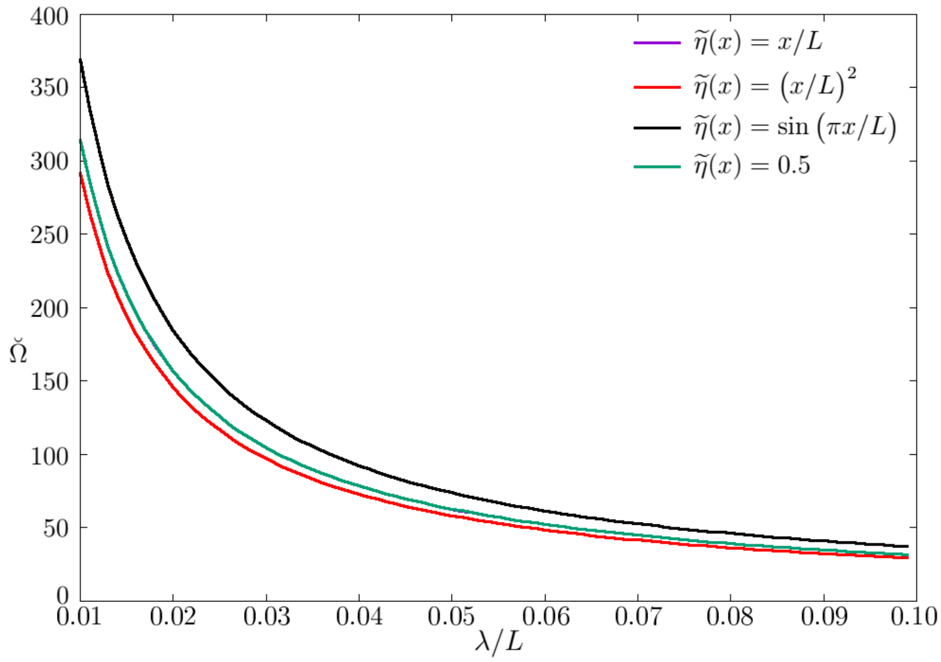

Figure 7.6: Diagrams of dimensionless free micro-vibration frequency $\breve{\Omega}$ (7.14) versus dimensionless microstructure length parameter λ/L, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$

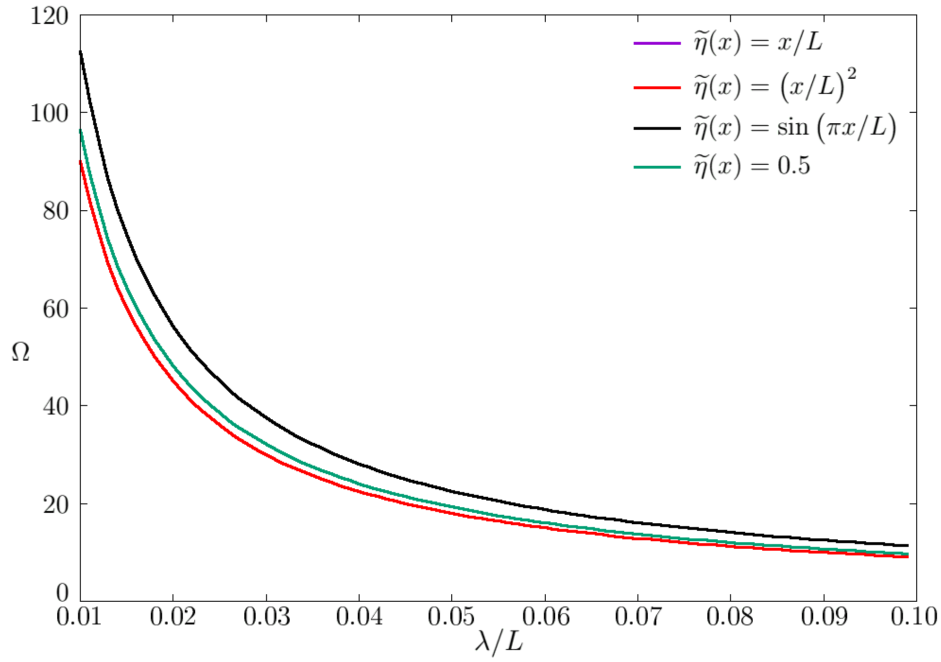

Figure 7.7: Diagrams of dimensionless free micro-vibration frequency Ω (7.15) versus dimensionless microstructure length parameter λ/L, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $E_2/E_1 = 0.25$, $\rho_2/\rho_1 = 0.75$

The cell-dependent higher free micro-vibration frequencies discussed here can be also determined applying the tolerance model equations (5.6), (5.7). However, equations (5.6), (5.7) are much more complicated than the microscopic tolerance equations (5.34)-(5.36) of the combined model, which are independent of solutions obtained in the framework of the asymptotic model derived in the first step of combined modeling. Moreover, within the tolerance model governed by equations (5.6), (5.7), these higher free vibration frequencies are always determined not separately but together with the fundamental cell-independent lower free vibration frequencies. In order to check this conformability, the transversal dimensionless free micro-vibration frequency Ω (7.15) obtained on the basis of micro-dynamic equation (5.36) of the combined model (for $A = N = 1$) will be compared with corresponding frequency $\widehat{\Omega}_+$ (6.57) derived in the framework of tolerance model (5.6), (5.7).

We recall that calculations based on (6.57) were carried out for a simply supported shell, for Poisson's ratio $\nu = 0.3$, $L_2/L_1 = 2/\pi$, $d/\lambda = 12/(125\pi)$, $\lambda/L_1 = 1/24$ and for wave numbers $\alpha = 5\pi/L_1$, $\beta = \pi/L_2$. In order to compare (7.15) and (6.57), calculations based on (7.15) will be made for the same data.

In Fig. 7.8 there are shown diagrams of higher free vibration frequency Ω (7.15) derived from the combined model versus both ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ described by (6.14)-(6.16).

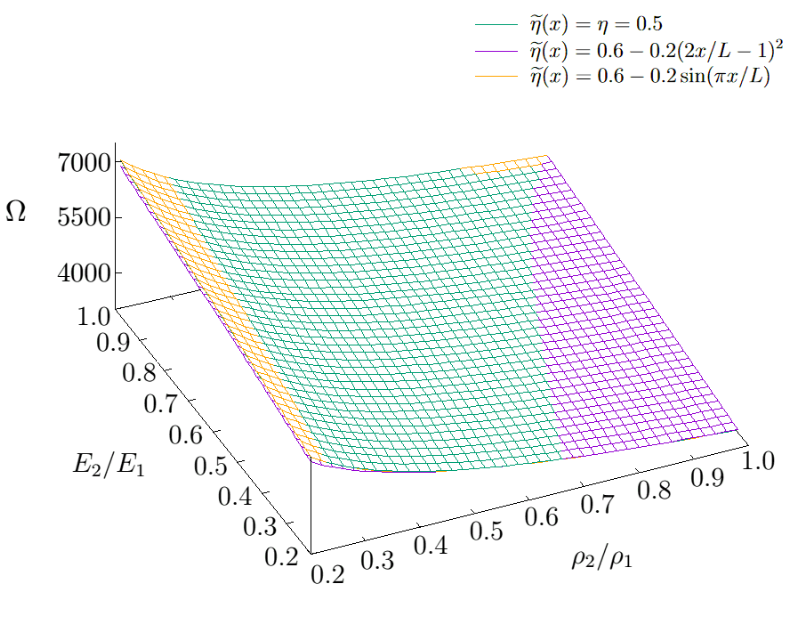

Figure 7.8: Diagrams of dimensionless free micro-vibration frequency Ω (7.15) versus ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.14)-(6.16) and for $\lambda/L = 1/24$, $d/\lambda = 12/(125\pi)$

It can be observed that diagrams shown in Figs. 7.8 and 6.24 are nearly identical. Maximum relative error between values of Ω (7.15) derived from the combined model and values of $\widehat{\Omega}_+$ (6.57) derived from the tolerance model for distribution functions (6.14)-(6.16) are equal to 0.0178%, 0.0172%, 0.0176%, respectively. In all cases, $\widehat{\Omega}_+$ (6.57) derived from the tolerance model was bigger than Ω (7.15) derived from the combined model.

7.2.3 Discussion of results and conclusions

On the basis of results shown in Figs. 7.2-7.8 the following conclusions can be formulated:

- The values of dimensionless free micro-vibration frequencies $\overline{\Omega}$, $\breve{\Omega}$ and Ω given by (7.13)-(7.15) for all distribution functions under consideration increase with the increase of ratio $E_2/E_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between elastic properties of the shell component materials, cf. Figs. 7.2-7.4. Because the value of Young's module E_1 for the stronger material is fixed then these differences decrease if values of E_2 tend to value of E_1.

- The values of dimensionless free micro-vibration frequencies $\overline{\Omega}$, $\breve{\Omega}$ and Ω given by (7.13)-(7.15) for all distribution functions under consideration decrease with the increase of ratio $\rho_2/\rho_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between inertial properties of the shell component materials, cf. Figs. 7.2-7.4. Because the value of Young's module ρ_1 for the stronger material is fixed then these differences decrease if values of ρ_2 tend to value of ρ_1.

- For every distribution function under consideration, the free micro-vibration frequency in circumferential direction is bigger than the free micro-vibration frequency in axial direction, and free micro-vibration frequency in axial direction is greater than the transversal free micro-vibration frequency, cf. Figs. 7.2-7.4.

- The highest value of dimensionless free micro-vibration frequency $\overline{\Omega}$ (7.13) is obtained for distribution function $\widetilde{\eta}(x) = \sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 1.0$, $\rho_2/\rho_1 = 0.2$), cf. Fig. 7.2.

- The lowest value of dimensionless free micro-vibration frequency $\overline{\Omega}$ (7.13) is obtained for distribution function $\widetilde{\eta}(x) = (x/L)^2$ and for pair of ratios ($E_2/E_1 = 0.2$, $\rho_2/\rho_1 = 1.0$), cf. Fig. 7.2.

- The highest value of dimensionless free micro-vibration frequency $\breve{\Omega}$ (7.14) is obtained for distribution function $\widetilde{\eta}(x) = \sin(\pi x/L)$ and for pair of ratios ($E_2/E_1 = 1.0$, $\rho_2/\rho_1 = 0.2$), cf. Fig. 7.3.

- The lowest value of dimensionless free micro-vibration frequency $\breve{\Omega}$ (7.14) is obtained for distribution function $\widetilde{\eta}(x) = (x/L)^2$ and for pair of ratios ($E_2/E_1 = 0.2$, $\rho_2/\rho_1 = 1.0$), cf. Fig. 7.3.

- The highest value of dimensionless free micro-vibration frequency Ω (7.15) is obtained for distribution function $\widetilde{\eta}(x) = (x/L)^2$ and for pair of ratios ($E_2/E_1 = 1.0$, $\rho_2/\rho_1 = 0.2$), cf. Fig. 7.4.

- The lowest value of dimensionless free micro-vibration frequency Ω (7.15) is obtained for distribution function $\widetilde{\eta}(x) = (x/L)^2$ and for pair of ratios ($E_2/E_1 = 0.2$, $\rho_2/\rho_1 = 1.0$), cf. Fig. 7.4.

- For fixed values of E_2/E_1 and ρ_2/ρ_1, the values of dimensionless free micro-vibration frequencies $\overline{\Omega}$, $\widehat{\Omega}$ and Ω given by (7.13)-(7.15), respectively, are exponentially decreasing with increasing value of λ/L_1, cf. Figs. 7.5-7.7.

- Validation of the models is confirmed by the very good agreement (relative error is smaller than $1.78 \cdot 10^{-4}$) in the comparison of the higher free vibration frequencies Ω_+ (6.57) derived from the tolerance model and the higher free vibration frequencies derived from the combined model Ω (7.15).

7.3. Space-boundary layer phenomena

In this subsection we shall investigate influence of a microstructure size on the shape of displacement micro-fluctuations in the open cylindrical transversally graded shell under consideration. The shell is described in detail in Subsection 6.1 and shown in Fig. 6.1. In this subsection, approximation $\widetilde{h}(x,z)$ of fluctuation shape functions $h(x)$ given by $\widetilde{h}(x,z) = \lambda \sin(2\pi z/\lambda)$, $z \in \Delta(x)$, $x \in \Omega_\Delta$, will be taken into account.

A special length-scale problem of harmonic micro-vibrations in axial direction will be studied. The analysis will be based on Eq. (5.35). It will be shown that Eq. (5.35) describes certain space-boundary layer phenomena strictly related to the specific form of boundary conditions imposed on fluctuation amplitude $Q_2(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$.

7.3.1 Analytical results

The problem of harmonic micro-vibrations in axial direction is studied on the basis of Eq. (5.35). For $a = n = 1$ this equation reduces to Eq. (7.5). In order to make the analysis more clear, below we recall this equation

$$\underline{\langle D^{2222}(h)^2\rangle}(x)\partial_{22}Q_2 - \left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle(x)Q_2 - \underline{\left\langle \mu\left(h\right)^2\right\rangle}(x)\ddot{Q}_2 = 0, \qquad x \in \Omega_\Delta. \tag{7.16}$$

The underlined terms in (7.16) depend on microstructure length parameter λ.

Setting $\breve{Q}_2(x, \overline{\xi}, t) \equiv Q_2(x, L_2\overline{\xi}, t)$, where $\overline{\xi} \equiv \xi/L_2 \in [0,1]$, $(x,t) \in \Omega \times \mathrm{I}$, we transform equation (7.16) to the following dimensionless form with respect to dimensionless argument $\overline{\xi}$

$$(L_2)^{-2}\,\underline{\langle D^{2222}(h)^2\rangle}(x)\partial_{22}\breve{Q}_2 - \left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle(x)\breve{Q}_2 - \underline{\left\langle \mu\left(h\right)^2\right\rangle}(x)\ddot{\breve{Q}}_2 = 0, \tag{7.17}$$

In order to investigate the problem of harmonic micro-vibrations in axial direction, we assume solution to Eq. (7.17) in the form

$$\breve{Q}_2\left(x, \overline{\xi}, t\right) = \breve{Q}^*\left(x, \overline{\xi}\right)\cos\left(\breve{\omega}t\right) \tag{7.18}$$

with $\breve{\omega}$ as a vibration frequency.

Under denotations

$$\breve{k}^2 = \frac{(L_2)^2\left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle}{\lambda^2\left\langle D^{2222}\overline{h}^2\right\rangle}, \qquad \breve{\omega}_*^2 = \frac{\left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle}{\lambda^2\left\langle \mu\left(\overline{h}\right)^2\right\rangle} \tag{7.19}$$

where $\overline{h} = \lambda^{-1}h$, Eq. (7.17) yields

$$\partial_{22}\breve{Q}^*\left(x, \overline{\xi}\right) - \breve{k}^2\left[1 - \left(\frac{\breve{\omega}}{\breve{\omega}_*}\right)^2\right]\breve{Q}^*\left(x, \overline{\xi}\right) = 0, \tag{7.20}$$

where $\breve{\omega}_*$ is referred to as the cell-depending free micro-vibration frequency. It can be shown that for every $x \in [0, L_1]$, averages $\left\langle D^{2222}\overline{h}^2\right\rangle$, $\left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle$, $\left\langle \mu\left(\overline{h}\right)^2\right\rangle$ are greater than zero; hence $\breve{k}^2 > 0$ and $\breve{\omega}_*^2 > 0$.

Eq. (7.20) has a dimensionless form with respect to argument $\overline{\xi}$. Argument $x \in [0, L_1]$ in (7.20) can be treated as a parameter and $\breve{Q}^*(x, \overline{\xi})$ is a slowly-varying function in x. For every fixed x, Eq. (7.20) can be treated as ordinary differential equation with respect to $\overline{\xi}$ having *constant coefficients*.

The boundary conditions are assumed in the form $\breve{Q}^*\left(x, \overline{\xi} = 0\right) = \breve{Q}_0(x)$, $\breve{Q}^*\left(x, \overline{\xi} = 1\right) = 0$, where $\breve{Q}_0$ is the known function slowly-varying in x.

For an arbitrary but fixed $x \in [0, L_1]$, the solutions $\breve{Q}^*\left(x, \overline{\xi}\right)$ to equation (7.20) depend on relations between vibrations frequencies $\breve{\omega}$ and $\breve{\omega}_*$. It means that micro-fluctuation amplitude $\breve{Q}_2\left(x, \overline{\xi}, t\right)$ given by (7.18) also depends on relations between $\breve{\omega}$ and $\breve{\omega}_*$.

The following special cases of micro-vibrations can be taken into account

1. If $\breve{\omega} = 0$ then
$$\breve{Q}^*\left(x, \overline{\xi}\right) = \breve{Q}_0(x) \exp\left(-\breve{k}\,\overline{\xi}\right) \tag{7.21}$$
and fluctuation amplitude $\breve{Q}_2(x, \overline{\xi}, t)$ is given by
$$\breve{Q}_2\left(x, \overline{\xi}, t\right) = \breve{Q}^*\left(x, \overline{\xi}\right) = \breve{Q}_0(x) \exp\left(-\breve{k}\,\overline{\xi}\right); \tag{7.22}$$
we deal with a stationary problem with strongly decaying micro-fluctuactions.

2. If $0 < \breve{\omega}^2 < \breve{\omega}_*^2$ and setting $\breve{k}_\omega^2 \equiv \breve{k}^2 \left[1 - \left(\frac{\breve{\omega}}{\breve{\omega}_*}\right)^2\right]$ then
$$\begin{aligned}\breve{Q}^*\left(x, \overline{\xi}\right) = \breve{Q}_0(x) \Big[&\exp\left(-\breve{k}_\omega\,\overline{\xi}\right)\left(1 - \exp(-2\breve{k}_\omega)\right)^{-1} + \\ &+ \exp(\breve{k}_\omega \overline{\xi})\left(1 - \exp(2\breve{k}_\omega)\right)^{-1}\Big]\end{aligned} \tag{7.23}$$
and fluctuation amplitude $\breve{Q}_2\left(x, \overline{\xi}, t\right)$ has the form
$$\begin{aligned}\breve{Q}_2\left(x, \overline{\xi}, t\right) &= \breve{Q}^*\left(x, \overline{\xi}\right)\cos(\breve{\omega}t) = \\ &= \breve{Q}_0(x)\Big[\exp\left(-\breve{k}_\omega\overline{\xi}\right)\left(1 - \exp\left(-2\breve{k}_\omega\right)\right)^{-1} + \\ &+ \exp\left(\breve{k}_\omega\overline{\xi}\right)\left(1 - \exp\left(2\breve{k}_\omega\right)\right)^{-1}\Big]\cos(\breve{\omega}t).\end{aligned} \tag{7.24}$$

In this case ***micro-vibrations decay exponentially***. It can be observed that if $0 < \breve{\omega}^2 \ll \breve{\omega}_*^2$ then we can take into account the following approximate form of solution (7.24)

$$\begin{aligned}\breve{Q}_2\left(x,\overline{\xi},t\right) &= \breve{Q}^*\left(x,\overline{\xi}\right)\cos(\breve{\omega}t) = \\ &= \breve{Q}_0\left(x\right)\exp\left(-\breve{k}_\omega\overline{\xi}\right)\cos\left(\breve{\omega}t\right).\end{aligned} \tag{7.25}$$

From (7.25) it follows that ***micro-vibrations are strongly decaying near the boundary*** $\overline{\xi} = 0$. It means that they can be treated as equal to zero outside a certain narrow layer near boundary $\overline{\xi} = 0$. Thus, equation (7.16) being a starting point in the micro-dynamic problem under consideration makes it possible to investigate ***the boundary-layer phenomena***.

3. If $\breve{\omega}^2 = \breve{\omega}_*^2$ then

$$\breve{Q}^*\left(x,\overline{\xi}\right) = \breve{Q}_0\left(x\right)\left(1-\overline{\xi}\right) \tag{7.26}$$

and fluctuation amplitude $\breve{Q}_2\left(x,\overline{\xi},t\right)$ is given by

$$\breve{Q}_2\left(x,\overline{\xi},t\right) = \breve{Q}^*\left(x,\overline{\xi}\right)\cos\left(\breve{\omega}t\right) = \breve{Q}_0\left(x\right)\left(1-\overline{\xi}\right)\cos\left(\breve{\omega}t\right); \tag{7.27}$$

we deal with a linear decaying of micro-vibrations.

4. If $\breve{\omega}^2 > \breve{\omega}_*^2$ and $\breve{\kappa}^2 \equiv \breve{k}^2\left[\left(\frac{\breve{\omega}}{\breve{\omega}_*}\right)^2 - 1\right] \neq (n\pi)^2$ then

$$\breve{Q}^*\left(x,\overline{\xi}\right) = \breve{Q}_0(x)\sin\left(\breve{\kappa}(1-\overline{\xi})\right)\left(\sin(\breve{\kappa})\right)^{-1} \tag{7.28}$$

and fluctuation amplitude $\breve{Q}_2\left(x,\overline{\xi},t\right)$ has the form

$$\begin{aligned}\breve{Q}_2\left(x,\overline{\xi},t\right) &= \breve{Q}^*\left(x,\overline{\xi}\right)\cos(\breve{\omega}t) = \\ &= \breve{Q}_0\left(x\right)\sin\left(\breve{\kappa}\left(1-\overline{\xi}\right)\right)\sin\left(\breve{\kappa}\right)^{-1}\cos(\breve{\omega}t);\end{aligned} \tag{7.29}$$

micro-vibrations are not decaying, they oscillate.

5. If $\breve{\omega}^2 > \breve{\omega}_*^2$ and setting $\breve{\kappa}^2 \equiv \breve{k}^2\left[\left(\frac{\breve{\omega}}{\breve{\omega}_*}\right)^2 - 1\right] = (n\pi)^2$ then the solution to equation (7.20) does not exist; ***we obtain resonance microvibrations with resonance frequencies***

$$\breve{\omega}_n^2 = \breve{\omega}_*^2\left[1 + \frac{(n\pi)^2}{\breve{\kappa}^2}\right], \qquad n = 1, 2\ldots \tag{7.30}$$

It has to be emphasized that *the above effect cannot be analysed in the framework of asymptotic models commonly used for investigations of dynamic problems for tolerance-periodic shells*. It can be observed that within the asymptotic models, equation (7.16) reduces to equation $\left\langle D^{2112} (\partial_1 h)^2 \right\rangle Q_2 = 0$, which has only trivial solution $Q_2 = 0$.

7.3.2 Numerical calculations

Plots of solutions $\breve{Q}^* \left(x, \bar{\xi}\right)$ to Eq. (7.20) given by (7.21), (7.23), (7.26), (7.28) are presented in Figs. 7.10-7.14. We recall that $\breve{Q}^* \left(x, \bar{\xi}\right)$ is a part of micro-fluctuation amplitude $\breve{Q}_2 \left(x, \bar{\xi}, t\right)$, i.e. $\breve{Q}_2 \left(x, \bar{\xi}, t\right) = \breve{Q}^* \left(x, \bar{\xi}\right) \cos(\breve{\omega} t)$.

Calculations are made for Poisson ratio $\nu = 0.3$, for fixed ratios $L_2/L_1 = 2$, $d/\lambda = 0.1$ and for various ratios $\varepsilon \equiv \lambda/L_1 \in [0.01, 0.1]$, $E_2/E_1 \in [0.2, 1]$. It can be observed that under assumption $L_2/L_1 = 2$, values of ratio $\varepsilon \equiv \lambda/L_1$ imply values of ratio λ/L_2, i.e. $\lambda/L_2 = \lambda/(2L_1) = 0.5\varepsilon$.

All diagrams are made for approximations $\widetilde{\eta}(x)$ of material properties distribution functions $\eta(x)$ given by (6.6), (6.7), (6.10), (6.17), i.e. for $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \sin(\pi x/L)$, $\widetilde{\eta}(x) = \eta = 0.5$. We recall that $L \equiv L_1$. Diagrams of these functions are shown in Fig. 6.3. Additionally, distribution of materials described by functions applied in this subsection are shown in Fig. 7.9.

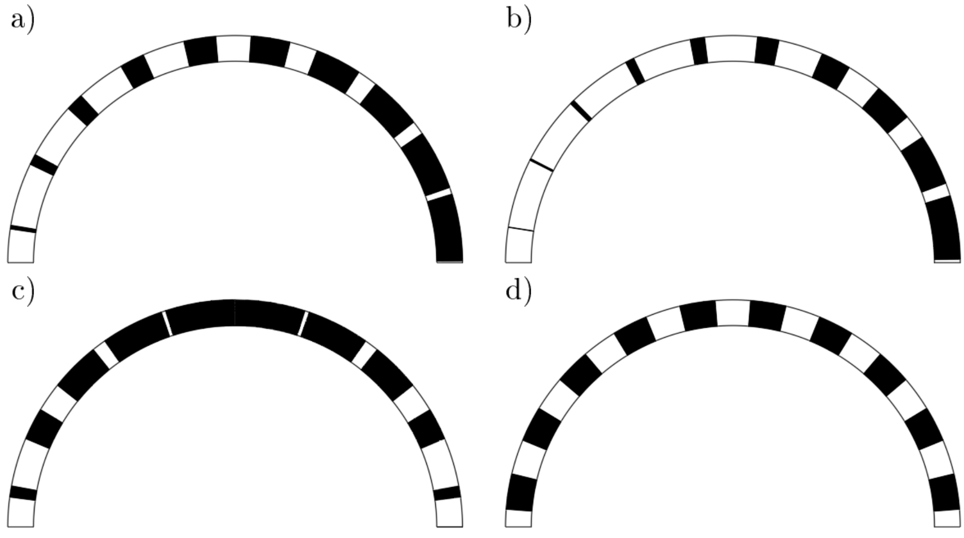

Figure 7.9: Distribution of materials described by a) $\widetilde{\eta}(x) = x/L$ b) $\widetilde{\eta}(x) = (x/L)^2$ c) $\widetilde{\eta}(x) = \sin(\pi x/L)$ d) $\widetilde{\eta}(x) = \eta = 0.5$

Plots for the exponentially decaying solutions $\breve{Q}^*((\breve{\omega}/\breve{\omega}_*)^2 < 1 : (\breve{\omega}/\breve{\omega}_*)^2 = 0.00, (0.65)^2, (0.80)^2)$ versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0, 0.1]$ and plots for exponentially and linearly decaying solutions $((\breve{\omega}/\breve{\omega}_*)^2 \leq 1 : \quad (\breve{\omega}/\breve{\omega}_*)^2 = 0.00, (0.65)^2, (0.80)^2, (0.90)^2, (0.98)^2, 1.00)$ versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0, 1]$ are shown in Figs. 7.10, 7.11. These diagrams are performed for ratios $\lambda/L_2 = 0.1$, $E_2/E_1 = 0.5$, $x/L_1 = 0.25$.

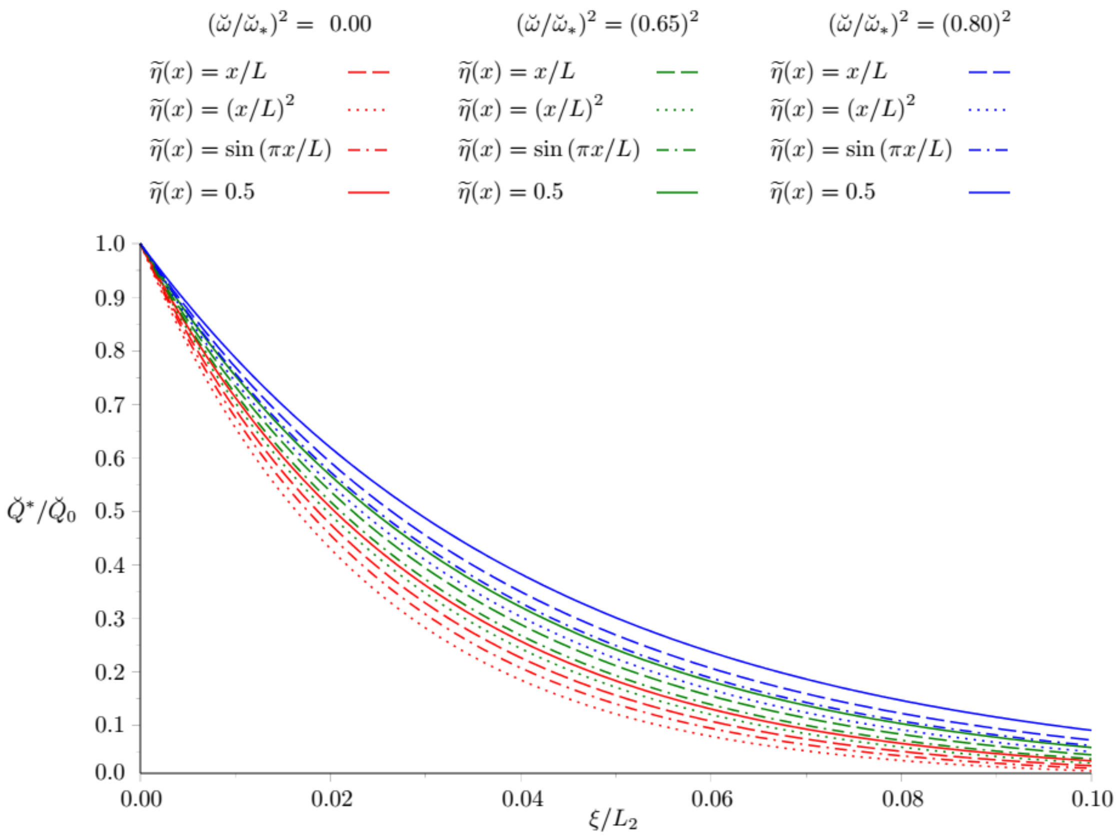

Figure 7.10: Diagrams of decaying solutions $\breve{Q}^*$ versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0, 0.1]$, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L_2 = 0.1$, $E_2/E_1 = 0.5$, $x = 0.25L_1$

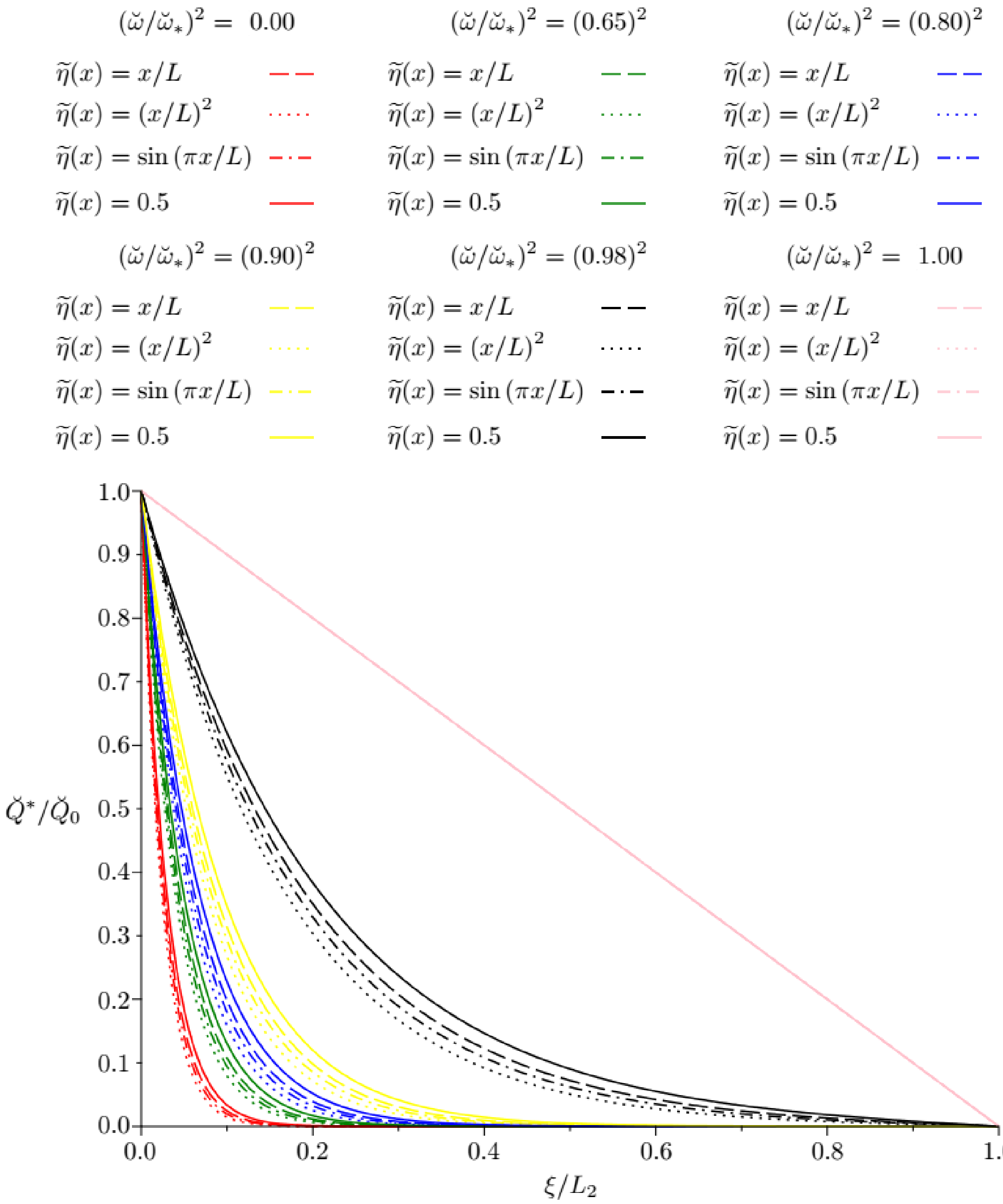

Figure 7.11: Diagrams of decaying solutions $\breve{Q}^*$ versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0,1]$, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L_2 = 0.1$, $E_2/E_1 = 0.5$, $x = 0.25L_1$

Plots for the oscillating solutions $\breve{Q}^*$ ($(\breve{\omega}/\breve{\omega}_*)^2 > 1 : (\breve{\omega}/\breve{\omega}_*)^2 = (1.1)^2$) versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0, 1]$ are presented in Fig. 7.12. These diagrams are performed for ratios $\lambda/L_2 = 0.1$, $E_2/E_1 = 0.5$, $x/L_1 = 0.25$.

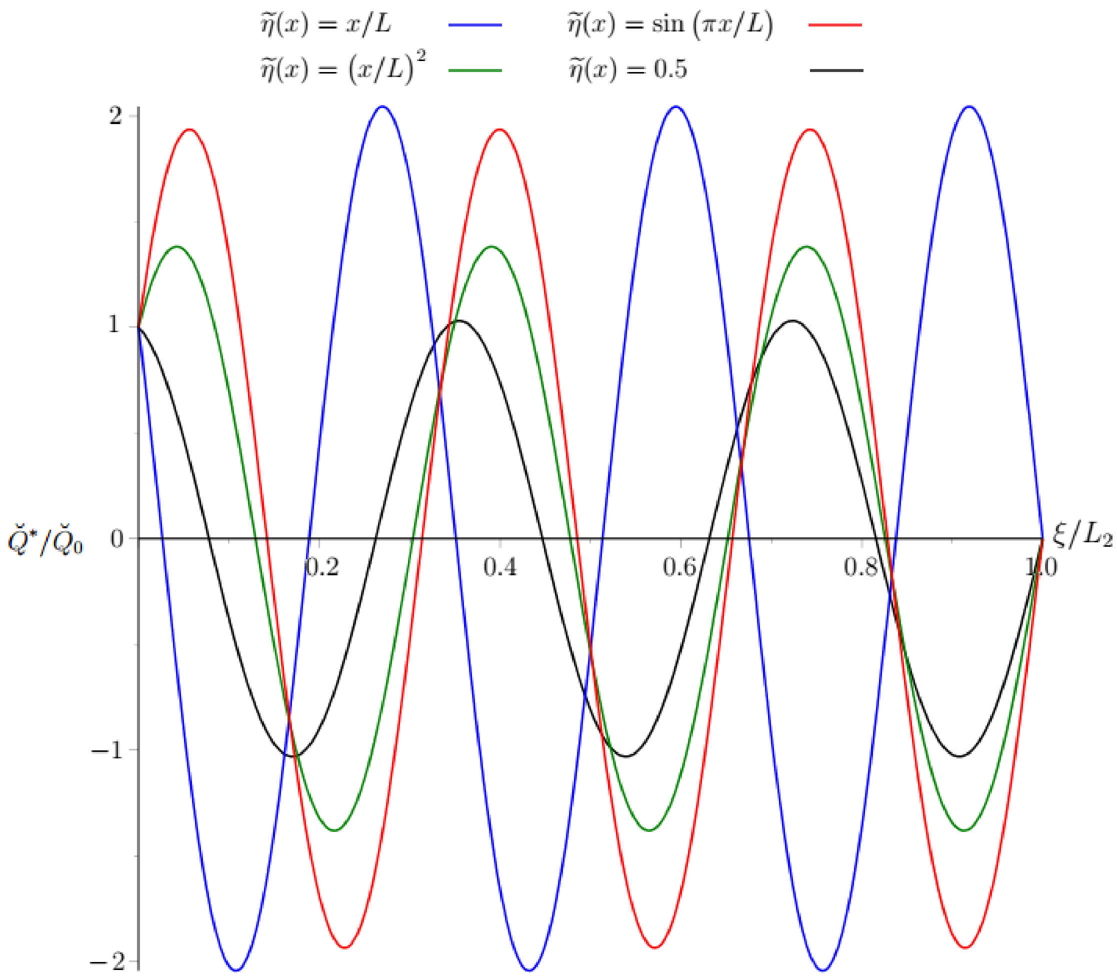

Figure 7.12: Diagrams of oscillating solutions $\breve{Q}^*$ versus dimensionless coordinate $\overline{\xi} \equiv \xi/L_2 \in [0, 1]$, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L_2 = 0.1$, $E_2/E_1 = 0.5$, $x = 0.25L_1$

In Fig. 7.13 there are diagrams of solutions $\breve{Q}^*(x,\overline{\xi})$ versus ratio $E_2/E_1 \in [0.2, 1]$ made for $\lambda/L_2 = 0.1$, $x/L_1 = 0.25$, $\overline{\xi} = 0.05$ and for $((\breve{\omega}/\breve{\omega}_*)^2 \leq 1 : (\breve{\omega}/\breve{\omega}_*)^2 = 0.00, (0.65)^2, (0.80)^2, (0.90)^2, (0.98)^2, 1.00)$.

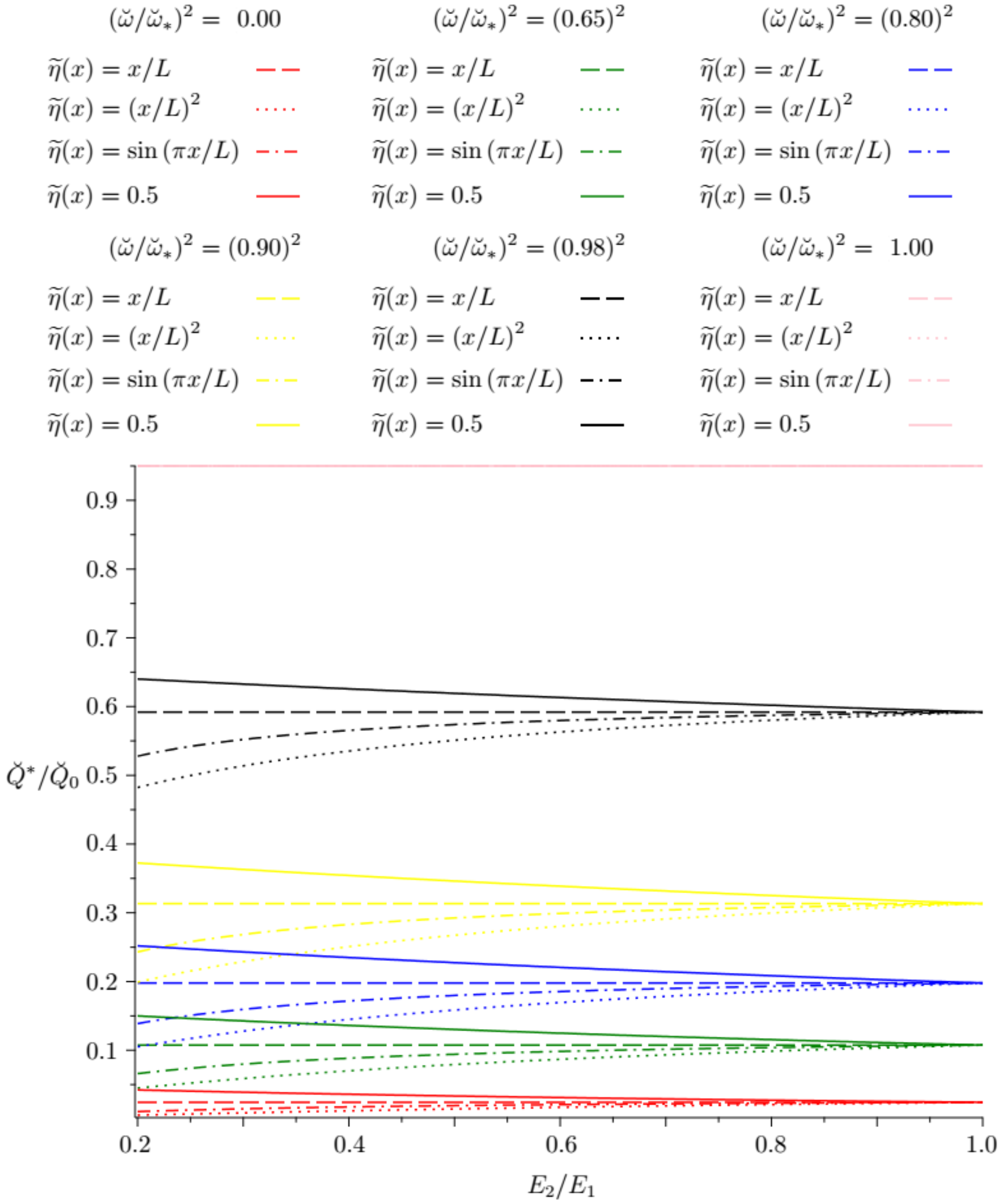

Figure 7.13: Diagrams of solutions $\breve{Q}^*$ versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $\lambda/L_2 = 0.1$, $x/L_1 = 0.25$, $\xi/L_2 = 0.05$

Plots of solutions $\breve{Q}^*(x,\overline{\xi})$ versus ratio $\lambda/L_2 \in [0.01, 0.1]$ performed for $E_2/E_1 = 0.5$, $x/L_1 = 0.25$, $\overline{\xi} = 0.05$ and for $((\breve{\omega}/\breve{\omega}_*)^2 \le 1 : (\breve{\omega}/\breve{\omega}_*)^2 = 0.00, (0.65)^2, (0.80)^2, (0.90)^2, (0.98)^2, 1.00)$ are shown in Fig. 7.14.

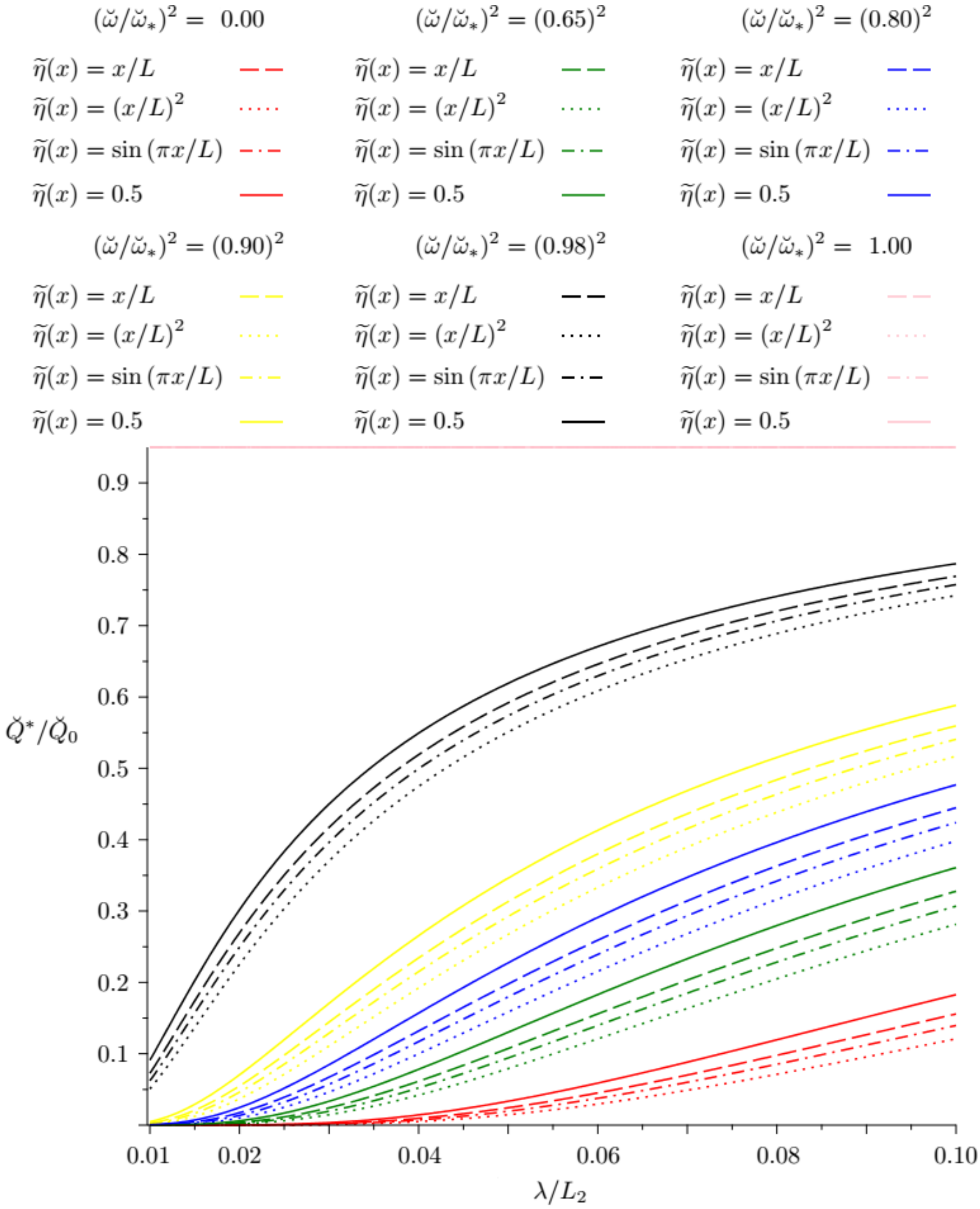

Figure 7.14: Diagrams of solutions $\breve{Q}^*$ versus ratio λ/L_2, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.10), (6.16) and for $E_2/E_1 = 0.5$, $x/L_1 = 0.25$, $\xi/L_2 = 0.05$

7.3.3 Discussion of results and conclusions

Analysing the obtained analytical and numerical results some remarks can be formulated:

- The shape of harmonic micro-vibrations with vibration frequency $\breve{\omega}$ depends on relations between values of $\breve{\omega}$ and *a certain new additional higher-order free vibration frequency $\breve{\omega}_*$ depending on a cell size λ*. ***The micro-vibrations can decay exponentially and very strongly near the boundary*** $\bar{\xi} = \xi/L_2 = 0$ and can be treated as equal to zero outside a certain narrow layer near this boundary, cf. solutions (7.22), (7.24), (7.25) and Figs. 7.10, 7.11. The problem with strongly decaying micro-vibrations near the boundary $\bar{\xi} = \xi/L_2 = 0$ is referred to ***the space-boundary-layer phenomena***. ***They can decay linearly, cf. solutions (7.27) and Fig. 7.11. Certain values of $\breve{\omega}$ cause a non-decayed form of micro-vibrations (micro-vibrations oscillate), cf. solution (7.29) and Fig. 7.12, for certain values of $\breve{\omega}$ we deal with resonance micro-vibrations, cf (7.30)***.

- From results presented in Figs. 7.10, 7.11 it follows that the mildest decrease in solutions $\breve{Q}^*$ takes place for distribution function $\widetilde{\eta}(x) = 0.5$, slightly stronger decrease is observed for $\widetilde{\eta}(x) = x/L$, the even stronger decrease takes place for $\widetilde{\eta}(x) = \sin(\pi x/L)$ and the strongest one is observed for $\widetilde{\eta}(x) = (x/L)^2$.

- From results shown in Fig. 7.13 it follows that solutions $\breve{Q}^*$ are almost constant for distribution function $\widetilde{\eta}(x) = x/L$, in contrast to distribution function $\widetilde{\eta}(x) = 0.5$, for which solutions $\breve{Q}^*$ decrease with the increase of E_2/E_1, and in contrast to distribution functions $\widetilde{\eta}(x) = \sin(\pi x/L)$ and $\widetilde{\eta}(x) = (x/L)^2$, for which solutions $\breve{Q}^*$ increase with the increase of E_2/E_1.

- Analysing results presented in Fig. 7.14 we can observe that the solutions $\breve{Q}^*$ increase with the increase of parameter λ/L_2 for all distribution functions and for all analysed values of $\breve{\omega}/\breve{\omega}_*$.

7.4. Wave propagation problem

In this subsection we shall analyse the long wave propagation problem for the tolerance-periodic shell under consideration. The shell is described in detail in Subsection 6.1 and shown in Fig. 6.1.

Let the considered shell be unbounded along the axial ξ-coordinate. We deal with long waves if condition $\lambda/\overline{L} \ll 1$ holds, where λ is the midsurface length parameter and $\overline{L}$ is the wavelength. The waves related to micro-fluctuation amplitude $Q_2(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, are taken into account. Hence, equation (5.35) describing the shell's micro-dynamics in an axial direction will be applied. For $a = n = 1$, this equation reduces to Eq. (7.16).

7.4.1 Analytical results

The problem of wave propagation in axial direction is studied on the basis of Eq. (7.16). We look for solution to (7.16) in the form $Q_2(x,\xi,t) = F(x,\xi - ct)$, where c is the wave propagation velocity. Setting $\overline{h} = \lambda^{-1}h$, from (7.16) we obtain

$$\left(c^2 - \tilde{c}^2\right)\partial_{22}F + \frac{\overline{c}^2}{\lambda^2}F = 0, \tag{7.31}$$

where, for an arbitrary but fixed $x \in \Omega$, speeds $\tilde{c}$ and $\overline{c}$ are defined by

$$\tilde{c}^2 = \frac{\left\langle D^{2222}\overline{h}^2\right\rangle}{\left\langle \mu\overline{h}^2\right\rangle}, \tag{7.32}$$

$$\overline{c}^2 = \frac{\left\langle D^{2112}\left(\partial_1 h\right)^2\right\rangle}{\left\langle \mu\overline{h}^2\right\rangle}. \tag{7.33}$$

It can be observed that argument x in (7.31) can be treated as a parameter and $F(x,\cdot)$, $x \in \Omega$, is a slowly-varying function in x. Equation (7.31) implies the following special cases of wave propagation in the tolerance-periodic shell under consideration:

1. sinusoidal waves if $c > \tilde{c}$,
2. exponential waves if $c < \tilde{c}$,
3. degenerate case if $c = \tilde{c}$.

The above effect cannot be analysed in the framework of asymptotic models.

In order to determine the dispersion relation for the case 1, let us substitute to (7.16) solution of the form $Q_2(x,\xi,t) = Af(x)\sin\left(k(\xi - ct)\right)$, $k = 2\pi/\overline{L}$, where $f(x)$ is the known slowly-varying function, $\overline{L}$ and k are the wavelength and the wave number, respectively, A is an arbitrary constant. It is assumed that $\overline{L} \gg \lambda$. The nontrivial solution ($A \neq 0$) exists only if

$$f(x)\left[(k\lambda)^2 c^2 - (k\lambda)^2\tilde{c}^2 - \overline{c}^2\right] = 0, \tag{7.34}$$

where under assumption that $\overline{L} \gg \lambda$, the following condition holds $k\lambda = 2\pi\lambda/\overline{L} \ll 1$.

The above equation describes the effect of dispersion. It can be seen that for $k\lambda \to 0$ the dispersion effect disappears. From (7.34) it follows that for a fixed $x \in \Omega$ such that $f(x) \neq 0$, the dispersive long waves related to micro-fluctuation amplitude $Q_2(x, \xi, t)$ can propagate across the unbounded tolerance-periodic shell under consideration with propagation speed

$$c^2 = \tilde{c}^2 + \frac{\overline{c}^2}{(k\lambda)^2} \tag{7.35}$$

depending on microstructure size λ.

7.4.2 Numerical calculations

All calculations are made using Maple by Maplesoft software and all charts are made in gnuplot.

For the shell under consideration, calculations are made for approximations $\widetilde{\eta}(x)$ of distribution functions of material properties $\eta(x)$ given by (6.6), (6.7), (6.11), (6.16), i.e. for $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = \cos(\pi x/(2L))$, $\widetilde{\eta}(x) = \eta = 0.5$. We recall $L \equiv L_1$.

Diagrams of these functions are shown in Fig. 6.3. Additionally, distribution of material properties described by functions applied in the problem analyzed here are shown in Fig. 7.15.

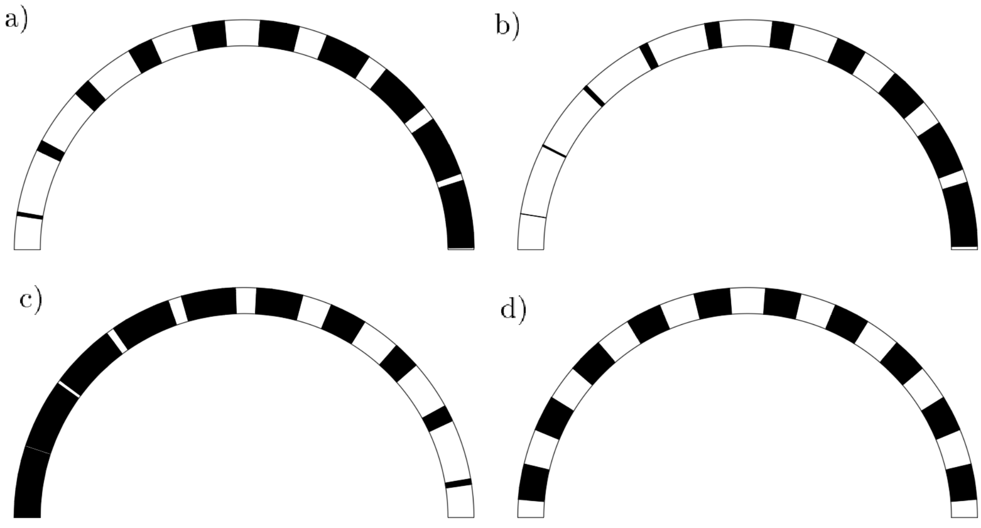

Figure 7.15: Distribution of materials described by a) $\widetilde{\eta}(x) = x/L$ b) $\widetilde{\eta}(x) = (x/L)^2$ c) $\widetilde{\eta}(x) = \cos(\pi x/(2L))$ d) $\widetilde{\eta}(x) = \eta = 0.5$

We define the following dimensionless wave propagation velocity

$$C^2 = \frac{\left(1 - \nu^2\right) \rho_1}{E_1} c^2 \tag{7.36}$$

where speed c is determined by formulae (7.35).

Calculations are made for fixed ratio $x/L_1 = 0.5$, for dimensionless wave number $k\lambda = 2\pi\lambda/L \in [0.01, 0.1]$ and for $E_2/E_1 \in [0.2, 1.0]$, $\rho_2/\rho_1 \in [0.2, 1.0]$.

From expressions (7.36) it follows that all plots are made under assumption $\nu = \text{const}$, $E_1 = \text{const}$, $\rho_1 = \text{const}$.

In Figs. 7.16-7.18 there are presented diagrams of dimensionless wave propagation velocity C given by (7.36) versus ratio ρ_2/ρ_1, made for distribution functions of material properties $\widetilde{\eta}(x)$ described by (6.6), (6.7), (6.11), (6.16) and for $E_2/E_1 = \{0.25, 0.5, 0.75\}$, $k\lambda = 0.02\ \pi$.

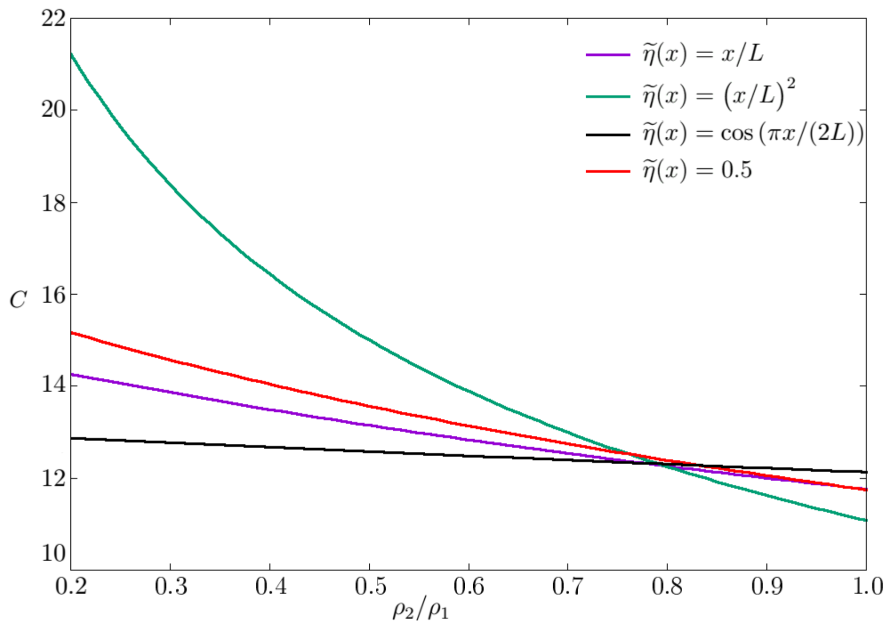

Figure 7.16: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $E_2/E_1 = 0.25$, $x/L_1 = 0.5$, $k\lambda = 0.02\ \pi$

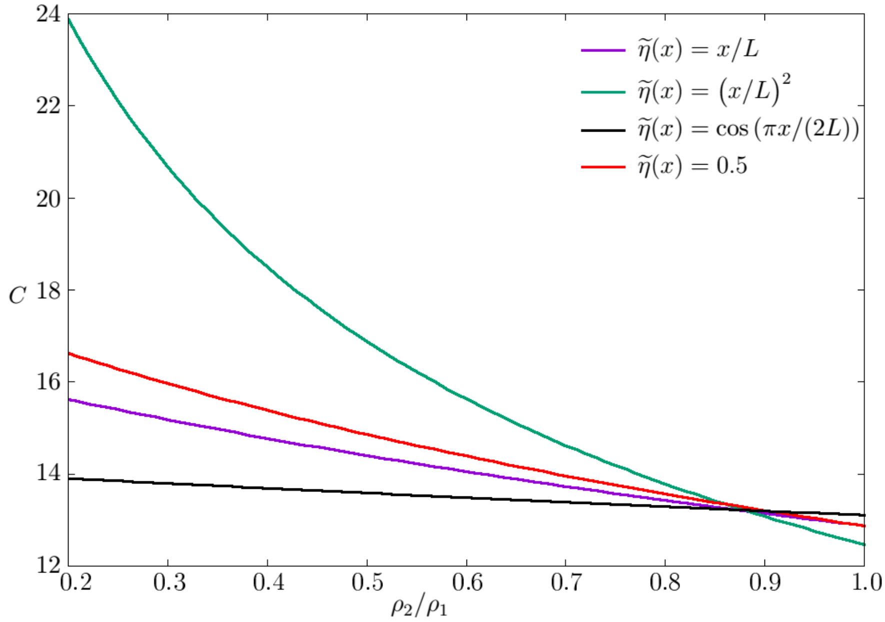

Figure 7.17: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $E_2/E_1 = 0.50$, $x/L_1 = 0.5$, $k\lambda = 0.02\ \pi$

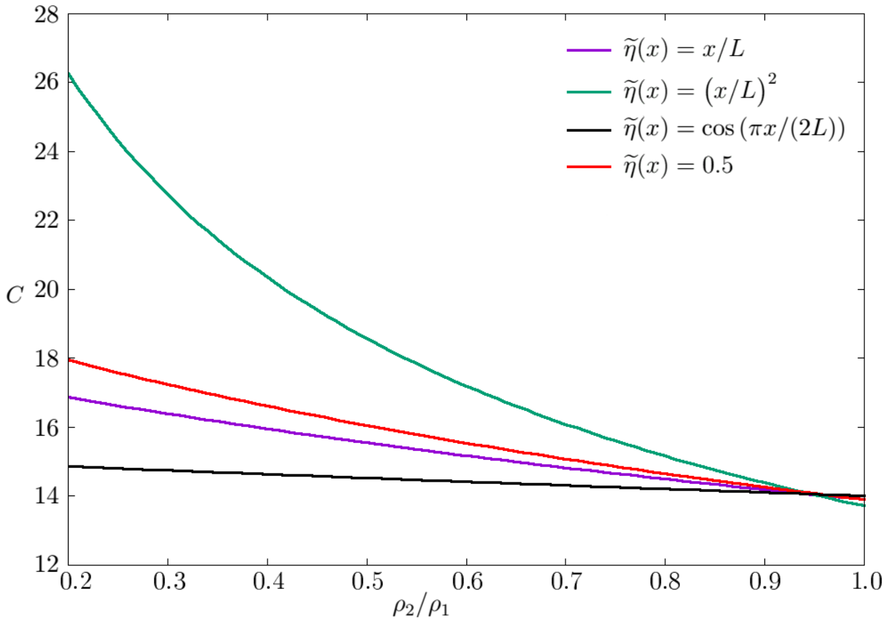

Figure 7.18: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $E_2/E_1 = 0.75$, $x/L_1 = 0.5$, $k\lambda = 0.02\ \pi$

In Figs. 7.19-7.21 there are presented diagrams of dimensionless wave propagation velocity C (7.36) versus ratio E_2/E_1, made for distribution functions (6.6), (6.7), (6.11), (6.16) and for $\rho_2/\rho_1 = \{0.25, 0.5, 0.75\}$, $k\lambda = 0.02\ \pi$.

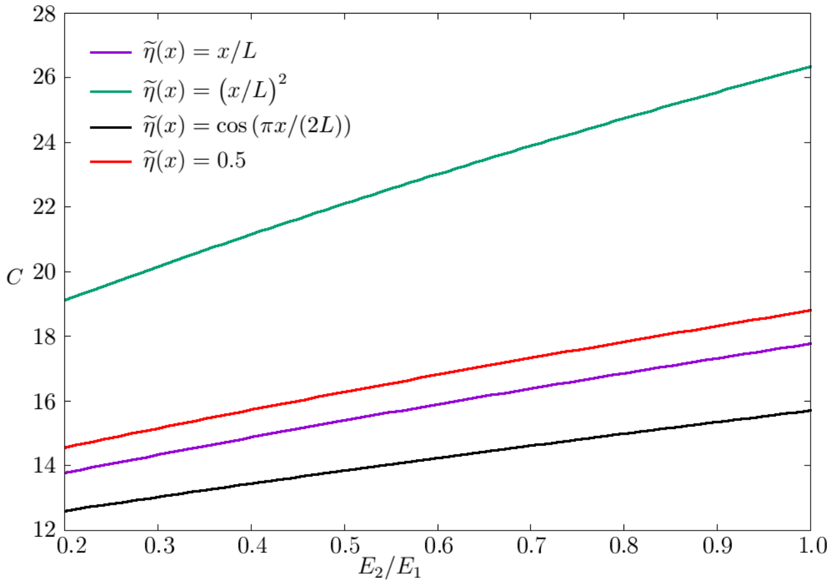

Figure 7.19: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $\rho_2/\rho_1 = 0.25$, $x/L_1 = 0.5$, $k\lambda = 0.02\ \pi$

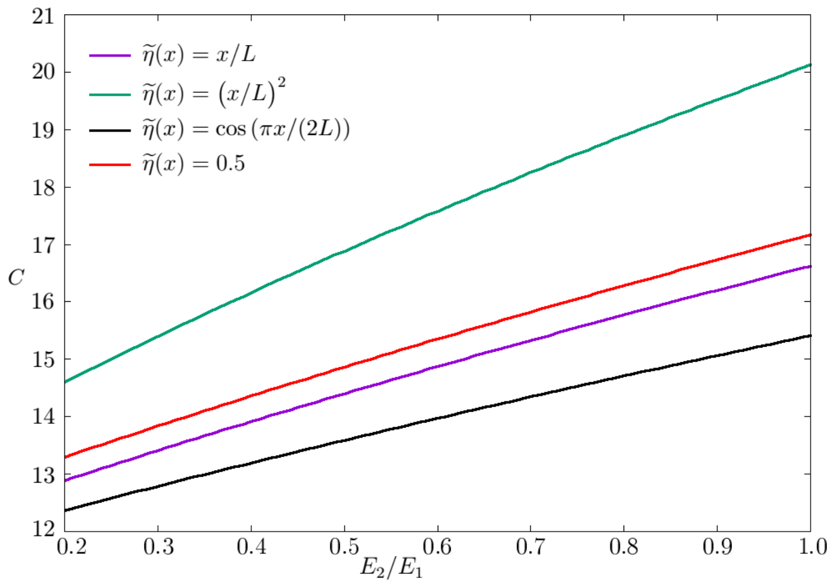

Figure 7.20: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $\rho_2/\rho_1 = 0.50$, $x/L_1 = 0.5$, $k\lambda = 0.02\ \pi$

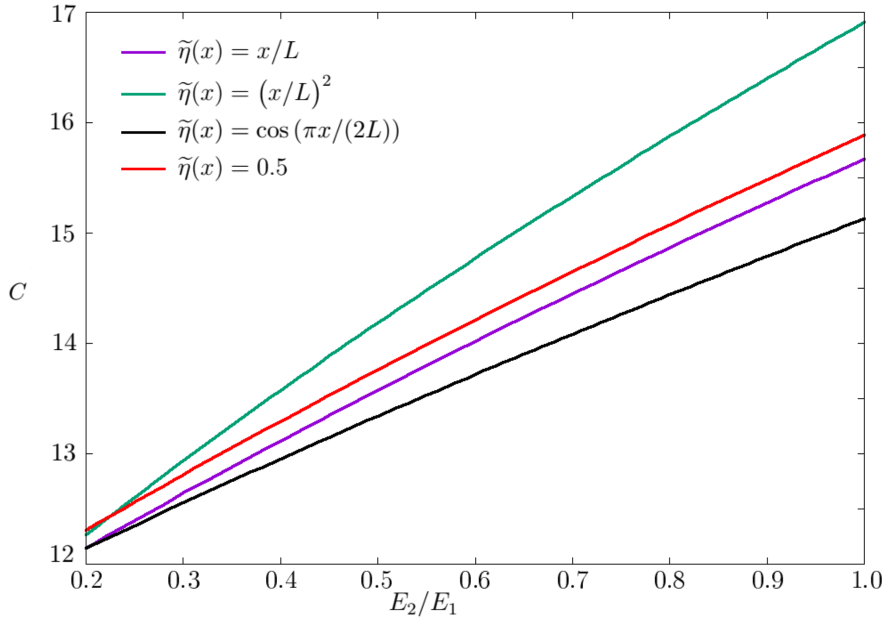

Figure 7.21: Diagrams of dimensionless wave propagation velocity C (7.36) versus ratio E_2/E_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $\rho_2/\rho_1 = 0.75$, $x/L_1 = 0.5$, $k\lambda = 0.02\,\pi$

In Fig. 7.22 there are presented diagrams of dimensionless wave propagation velocity C given by (7.36) versus dimensionless wave number $k\lambda$, made for distribution functions of material properties $\widetilde{\eta}(x)$ described by (6.6), (6.7), (6.11), (6.16) and for $\rho_2/\rho_1 = 0.25$, $E_2/E_1 = 0.25$.

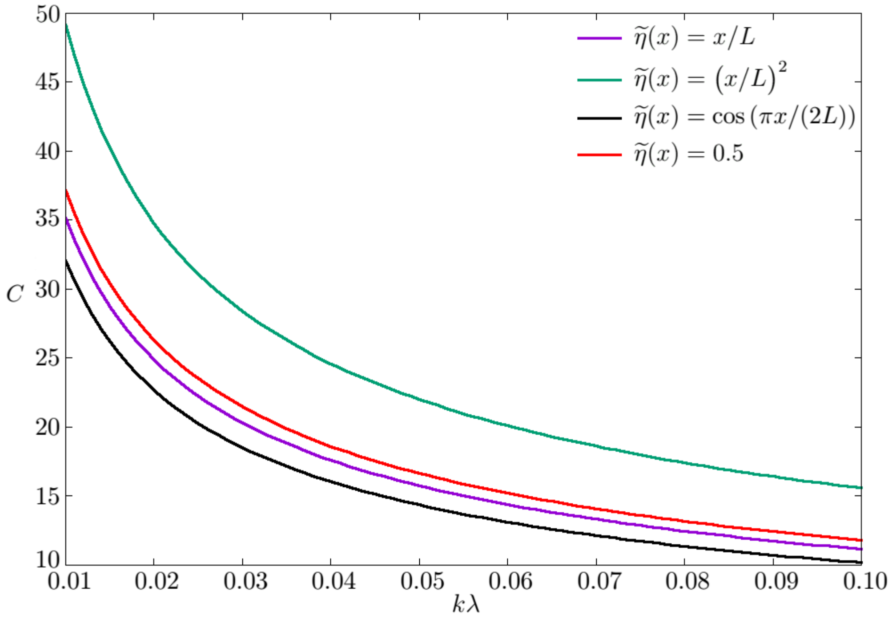

Figure 7.22: Diagrams of dimensionless wave propagation velocity C (7.36) versus dimensionless wave number $k\lambda$, made for distribution functions $\widetilde{\eta}(x)$ given by (6.7), (6.8), (6.11), (6.16) and for $x/L_1 = 0.5$, $\rho_2/\rho_1 = 0.25$, $E_2/E_1 = 0.25$

7.4.3 Discussion of results and conclusions

It was shown that ***the tolerance-periodic heterogeneity of the shell under consideration leads to exponential waves and to dispersion effects, which cannot be analysed in the framework of the asymptotic models for periodic or tolerance-periodic shells***. Moreover, ***the new wave propagation speed depending on the microstructure size has been obtained***, cf. formula (7.35).

On the basis of results shown in Figs. 7.16-7.22, the following conclusions can be formulated:

- Values of dimensionless wave propagation velocity C (7.36) decrease with the increasing of ratio $\rho_2/\rho_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between inertial properties of the component materials, cf. Figs. 7.16-7.18. Because the value of mass density ρ_1 for the stronger material is fixed then these differences decrease if values of ρ_2 tend to value of ρ_1.

- Values of dimensionless wave propagation velocity C (7.36) increase with the increasing of ratio $E_2/E_1 \in [0.2, 1.0]$, i.e. with the decreasing of differences between elastic properties of the shell component materials, cf. Fig. 7.19-7.21. Because the value of Young's module E_1 for the stronger material is fixed then these differences decrease if values of E_2 tend to value of E_1.

- Values of dimensionless speed C (7.36) decrease with the increase of dimensionless wave number $k\lambda$, i.e. with the decreasing of differences between microstructure length parameter λ and the wavelength $\overline{L}$, cf. Fig. 7.21. The strongest decrease in the dimensionless speed C takes place for $k\lambda \in [0.01, 0.03]$.

- The highest values of dimensionless speed C, cf. Figs. 7.16-7.21, are obtained for pair of ratios $(E_2/E_1 = 1.0, \rho_2/\rho_1 = 0.25)$, i.e. for a tolerance-periodic shell with a very strong inertial heterogeneity and with elastic homogeneous structure, and for distribution function $\widetilde{\eta}(x) = (x/L)^2$. The smallest values of this speed is obtained for pair of ratios $(E_2/E_1 = 0.25, \rho_2/\rho_1 = 1.0)$, i.e. for a tolerance-periodic shell with a very strong elastic heterogeneity and with inertial homogeneous structure, and for distribution function $\widetilde{\eta}(x) = (x/L)^2$.

- From results shown in Figs. 7.16-7.18 it follows that for values of ratio ρ_2/ρ_1 smaller than 0.8, Fig. 7.16, 0.87, Fig. 7.17, 0.94, Fig. 7.18, the values of dimensionless wave propagation speed C (7.36) are greatest for distribution function $\widetilde{\eta}(x) = (x/L)^2$, slightly smaller for $\widetilde{\eta}(x) = 0.5$, even smaller for $\widetilde{\eta}(x) = x/L$ and the smallest for $\widetilde{\eta}(x) = \cos(\pi x/(2L))$. On the other hand, for values of ratio ρ_2/ρ_1 greater than 0.8, Fig. 7.16, 0.87, Fig. 7.17, 0.94, Fig. 7.18, the order is reversed, i.e. the values of dimensionless velocity C (7.36) are greatest for distribution function $\widetilde{\eta}(x) = \cos(\pi x/(2L))$, slightly smaller for $\widetilde{\eta}(x) = x/L$, even smaller for $\widetilde{\eta}(x) = 0.5$ and the smallest for $\widetilde{\eta}(x) = (x/L)^2$.

7.5. Special length-scale initial value problem

Object of considerations is a thin shell strip with span $L \equiv L_1$ along the circumferential $x \equiv x^1$-coordinate and with a constant thickness. The shell strip has a tolerance-periodic microstructure and a functionally graded macrostructure along its span as well as a constant structure in the axial direction. It assumed that the shell strip is made of two elastic isotropic materials, which are perfectly bonded on interfaces and densely, tolerance-periodically distributed along x-coordinate. A fragment of such a shell strip is shown in Fig. 6.1, where in the problem under consideration length dimension L_2 of the shell along $\xi \equiv x^2$-coordinate is assumed to be infinite. The basic cell, properties of component materials and the shell rigidities are described in detail in Subsection 6.1.

The influence of microstructure length parameter λ on the character of displacement micro-fluctuations will be analysed in a certain special initial value problem. This problem will be treated as independent of the ξ-coordinate.

The analysis will be based on Eq. (5.34). For $a = n = 1$ and under assumption that the problem is independent of argument $\xi \in \Xi$, equation (5.34) reduces to the following form

$$\left\langle D^{1111} \left(\partial_1 h\right)^2 \right\rangle (x) Q_1(x,t) - \left\langle \mu(h)^2 \right\rangle (x) \ddot{Q}_1(x,t) = 0, \quad (x,t) \in \Omega \times \mathrm{I} \tag{7.37}$$

Argument $x \in \Omega$ in (7.37) can be treated as a parameter. For an arbitrary but fixed x, Eq. (7.37) can be treated as ordinary differential equation with respect to argument t having constant coefficients.

Introducing non-dimensional time coordinate $\tau = t/T$, where T is a certain time constant, defining function $q \equiv q_1(x,\tau) = Q_1(x,T\tau)$ and introducing parameter $\chi^2 \equiv (l/\lambda)^2$, where l is a certain length parameter given by

$$l^2 \equiv \left\langle D^{1111} \left(\partial_1 h\right)^2 \right\rangle T^2 \left\langle \mu(\overline{h})^2 \right\rangle^{-1} \tag{7.38}$$

with $\overline{h} \equiv \lambda^{-1} h$ (l is independent of a cell size λ) as well as denoting $\partial_\tau \equiv \partial/\partial_\tau$, we shall transform equation (7.37) to the dimensionless form with respect to time argument τ

$$\partial_{\tau\tau} q(x,\tau) + \chi^2 q(x,\tau) = 0. \tag{7.39}$$

For an arbitrary but fixed $x \in \Omega$, coefficient χ^2 in (7.39) is constant.

We shall assume the initial conditions in the form $q(x,0) = 1$, $\partial_\tau q(x,0) = 0$. Moreover the considerations will be restricted to the time interval $\tau \in [0,\pi/2]$. For an arbitrary but fixed $x \in \Omega$ and under initial conditions given above as well as for $\tau \in [0,\pi/2]$, the solutions to equation (7.39) are given in Fig. 7.23. ***These solutions illustrate the influence of microstructure length parameter λ on the character of micro-fluctuations in the circumferential direction.***

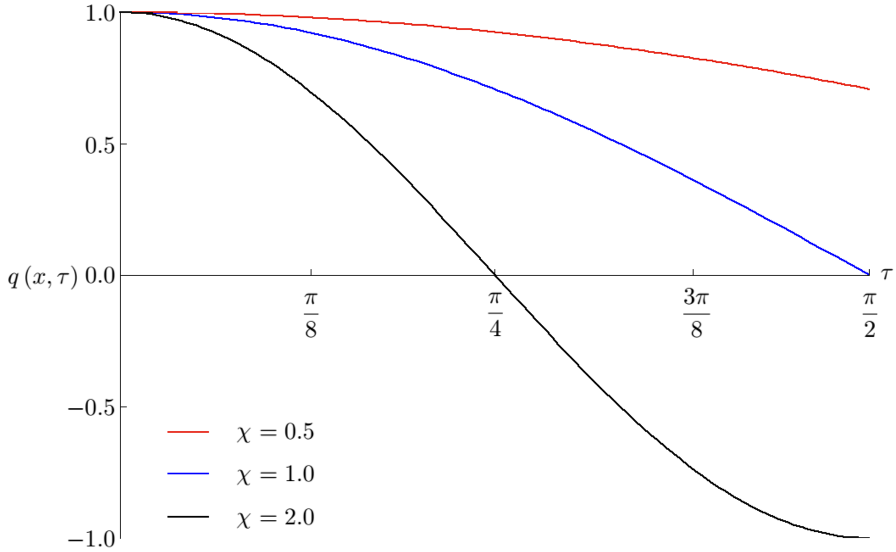

Figure 7.23: Diagrams of solutions to equation (7.39) versus dimensionless time coordinate τ, made for different interrelations between length parameters l and λ; $\chi \equiv l/\lambda$

On the basis of result shown in Fig. 7.23, the important conclusions are:

- if $0 < \chi < 1$, i.e. $l < \lambda$, then the displacement micro-fluctuations decrease monotonically and very softly; they don't take the zero-value in the time interval under consideration,
- if $\chi = 1$, i.e. $l = \lambda$, then the micro-fluctuations decay monotonically in the time interval under consideration; for $\tau = \pi/2$ they are equal to zero,
- if $\chi > 1$, i.e. $l > \lambda$, then solutions to equation (7.38) decay monotonically and strongly in a certain subinterval of the time interval under consideration, and then the absolute values of these solutions increase monotonically in the remaining part of this time interval.

At the end of this subsection, the influence of differences between elastic and inertial properties of the component materials on the length parameter l will be studied.

We introduce dimensionless length parameter

$$(L_d)^2 = \frac{\rho_1(1-\nu^2)}{E_1 T^2} l^2. \tag{7.40}$$

where l^2 is given by (7.38).

Calculations are made for Poisson ratio $\nu = 0.3$, for fixed ratios $d/L = 0.005$, $x/L = 0.25$, for various ratios $E_2/E_1 \in [0.2, 1]$ and $\rho_2/\rho_1 \in [0.2, 1]$ and for material properties distribution functions given by (6.6), (6.7), (6.9) and (6.16), i.e. for $\widetilde{\eta}(x) = x/L$, $\widetilde{\eta}(x) = (x/L)^2$, $\widetilde{\eta}(x) = (x/L)^3$, $\widetilde{\eta}(x) = \eta = 0.5$.

Diagrams of dimensionless length parameter L_d (7.40) versus both ratios E_2/E_1 and ρ_2/ρ_1 made for distribution functions $\widetilde{\eta}(x)$ given by (6.6), (6.7), (6.9) and (6.16) are presented in Fig. 7.24.

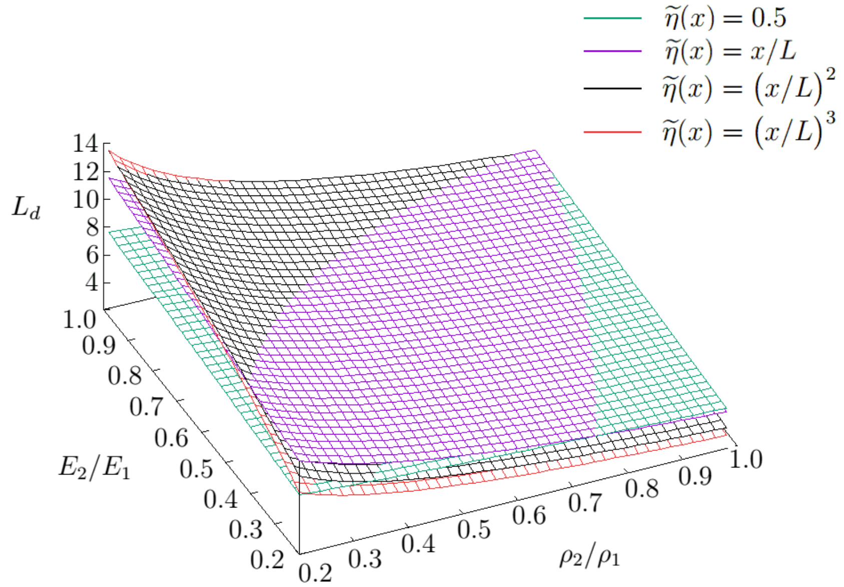

Figure 7.24: Diagrams of dimensionless length parameter L_d (7.40) versus both ratios E_2/E_1 and ρ_2/ρ_1, made for distribution functions $\widetilde{\eta}(x)$ given by (6.6), (6.7), (6.9), (6.16) and for $d/L = 0.005$, $x/L = 1/4$

From results shown in Fig. 7.24 it follows that dimensionless length parameter L_d (7.40) decreases with the increase of ratio ρ_2/ρ_1, i.e. with the decrease of differences between inertial properties of the component materials, but it increases with the increasing of ratio E_2/E_1, i.e. with the decreasing of differences between elastic properties of the component materials.

The highest value of dimensionless length parameter L_d (7.40) is obtained for pair of ratios ($E_2/E_1 = 1, \rho_2/\rho_1 = 0.2$), i.e. for a tolerance-periodic shell with a very strong inertial heterogeneity and with elastic homogeneous structure, and for distribution function $\widetilde{\eta}(x) = (x/L)^3$. The smallest value of this length parameter is obtained for pair of ratios ($E_2/E_1 = 0.2, \rho_2/\rho_1 = 1$), i.e. for a tolerance-periodic shell with a very strong elastic heterogeneity and with inertial homogeneous structure, and also for distribution function $\widetilde{\eta}(x) = (x/L)^3$.

It must be emphasized that the ***initial value problem discussed above cannot be described in the framework of the asymptotic models commonly used for investigations of mechanical problems for periodic or tolerance-periodic shells***.

8. Final remarks and conclusions

The objects of considerations are thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells having a functionally graded macrostructure and a tolerance-periodic microstructure in circumferential direction. ***On the microscopic level***, the shells consist of a very large number of separated, small elements regularly distributed along circumferential direction and perfectly bonded to each other (or to the homogeneous matrix). These elements, called ***cells***, are treated as thin shells. It is assumed that the ***adjacent cells are nearly identical*** (*i.e. they have nearly the same geometrical, elastic and inertial properties*), ***but the distant elements can be very different.*** The length dimension of a cell in circumferential direction, called ***the microstructure length parameter***, is assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the length dimension of the shell midsurface in the direction of tolerant periodicity. Examples of such shells are shown in Figs. 4.1 and 4.2. At the same time, the shells have constant structure in axial direction. ***On the microscopic level***, the geometrical, elastic and inertial properties of these shells are determined by highly oscillating, non-continuous and *tolerance-periodic functions* in circumferential direction. On the other hand, ***on the macroscopic level***, the averaged properties of the shells are described by functions being ***continuous and slowly varying*** along circumferential direction. It means that the tolerance-periodic shells under consideration can be treated as made of ***functionally graded materials*** (FGM), cf. Suresh and Mortensen [110], and called ***functionally graded shells.*** Moreover, since macroscopic properties of the shells are graded in direction normal to interfaces between constituents, this gradation is referred to as ***the transversal gradation.***

The subject-matter of this doctoral thesis is the analytical modelling of dynamic problems for the shells under consideration and the study of the effect of a cell size on the macroscopic and microscopic shell behaviour (*the length-scale effect*).

Dynamic behaviour of such shells are described by Euler-Lagrange equations (4.9) generated by Lagrange function (4.8). The explicit form of (4.9), given by

(4.10), coincides with the known governing equations of Kirchhoff-Love theory for thin elastic shells. For tolerance-periodic shells, coefficients of these equations are highly oscillating, non-continuous and tolerance-periodic functions. That is why the direct application of equations (4.10) to investigations of specific problems is non-effective even using computational methods.

The first aim of this dissertation was to formulate and discuss three new mathematical averaged models with continuously slowly-varying coefficients constituting a proper tool for the analysis of selected dynamic problems in the thin cylindrical shells with a tolerance-periodic microstructure and a transversally graded macrostructure in the circumferential direction. Moreover, two from these models take into account the effect of a microstructure size on the dynamic shell behaviour.

This aim has been realized by means of applying the tolerance, consistent asymptotic and combined asymptotic-tolerance modelling procedures, cf. [164] to the starting Euler-Lagrange equations (4.9), which explicit form (4.10) coincides with the known governing equations of Kirchhoff-Love theory for thin elastic shells.

The results can be summarized by the following remarks and conclusions:

a) ***The new mathematical non-asymptotic tolerance model*** for the analysis of selected dynamic problems in the functionally graded shells under consideration has been formulated by applying *the tolerance modelling procedure* discussed by Woźniak in a large number of contributions and summarized in [164, 166, 168.]. The *tolerance approach* is based on the notion of ***tolerance relations*** between points and real numbers related to the accuracy of the performed measurements and calculations. Tolerance relations are determined by ***tolerance parameters.*** Other fundamental concepts of this modelling technique are those of ***slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and averaging operation.*** Following monographs by Woźniak *et al.* (eds.) [164] and Ostrowski [90], the definitions of these basic notions were outlined in Chapter 3 of this dissertation. The fundamental assumption imposed on the lagrangian under consideration in the framework of *the tolerance averaging approach* is called ***the micro-macro decomposition.*** It states that the displacement fields occurring in this lagrangian have to be *the tolerance-periodic functions* in the direction of tolerant periodicity. Hence, they can be decomposed into ***unknown averaged displacements being slowly-varying functions and fluctuations represented by finite series of products of the known highly oscillating continuous***

tolerance-periodic fluctuation shape functions and unknown slowly-varying fluctuation amplitudes. *The second fundamental assumption is called* ***the tolerance averaging approximation****, cf.* (3.6). This assumption makes it possible to neglect terms of an order of tolerance parameters. The tolerance modelling technique used for Euler-Lagrange equations (4.9) is realized in two steps. The first step is based on ***the tolerance averaging of lagrangian*** (4.8) ***under micro-macro decomposition*** defined by (5.1). This averaging is realized by applying ***the averaging operation*** (3.5) and ***the tolerance averaging approximation*** (3.6). The resulting tolerance-averaged form of lagrangian (4.8) is given by (5.3). In the second step, applying *the principle of stationary action* to ***the tolerance-averaged action functional*** (5.4) determined by means of averaged lagrangian (5.3), we arrive at Euler-Lagrange equations (5.5). After combining (5.5) with (5.3), we obtain finally the explicit form of ***the tolerance model equations for the transversally graded shells under consideration.*** These equations are written in the form of constitutive relations (5.6) and dynamic balance equations (5.7). Unknowns of this model are ***macrodisplacements*** u^0_α, w^0 and ***fluctuation amplitudes*** U^a_α, W^A, $a = 1, 2, \ldots, n$, $A = 1, 2, \ldots, N$. These unknowns must be slowly-varying functions in the tolerant periodicity direction, i.e. they have to satisfy conditions (3.1) for the pertinent tolerance parameters. Contrary to starting equations (4.10) with coefficients highly oscillating, non-continuous and tolerance-periodic along x-coordinate parametrizing the shell midsurface in circumferential direction, ***the obtained tolerance model equations have coefficients which are continuous and slowly-varying in the direction of tolerant periodicity. Moreover, some of these coefficients depend on microstructure length parameter*** λ. It means that ***the biggest advantage of the proposed tolerance model is that it describes the effect of the cell size on the global shell behaviour.*** Moreover, ***this effect can be analysed not only in dynamic but also in stationary shell problems.*** It is worth mentioning that the tolerance equations proposed in this dissertation are a certain generalization of those derived and discussed in Tomczyk and Szczerba [146], which have been formulated under extra assumption $1 + \lambda/r \approx 1$, where λ and r stand respectively for the microstructure length parameter and the midsurface curvature radius. It means, that in the model equations presented here, terms of an order $O\big(\lambda/r\big)$ are not neglected and hence they contain a bigger number of terms depending on the microstructure size.

b) ***The new mathematical consistent asymptotic model*** for the analysis of selected dynamic problems in the transversally graded shells under consideration have been formulated by applying ***the new consistent asymptotic modelling procedure*** given in Woźniak *et al.* (eds.) [164]. The fundamental assumption imposed on the lagrangian under consideration in the framework of this approach is called ***the consistent asymptotic decomposition.*** It states that the displacement fields occurring in the lagrangian have to be replaced by families of fields depending on parameter $\varepsilon \in (0,1]$ and defined in an arbitrary cell. These families of displacements are decomposed into averaged part described by unknown functions being continuously bounded in the tolerant periodicity direction and highly oscillating part depending on ε. This highly oscillating part is represented by the known highly oscillating fluctuation shape functions multiplied by unknown functions being continuously bounded in the direction of tolerant periodicity. Asymptotic modelling procedure used for Euler-Lagrange equations (4.9) is realized in two steps. The first step is ***the consistent asymptotic averaging of lagrangian (4.9) under consistent asymptotic decomposition defined by (5.8).*** The resulting asymptotically averaged form of lagrangian (4.9) is given by (5.12). In the second step, applying *the principle of stationary action* to *the consistent asymptotic action functional* (5.13) defined by means of averaged lagrangian (5.12), we arrive at Euler-Lagrange equations (5.14). After combining (5.14) with (5.12), we obtain finally the explicit form of ***the consistent asymptotic model equations for the shells under consideration*** given by (5.15). Similarly as in the tolerance model, unknowns of the asymptotic one are called ***macrodisplacements and fluctuation amplitudes.*** However, these unknowns are not assumed to be slowly-varying functions satisfying conditions (3.1). They are assumed to be continuous and bounded in $\overline{\Omega} \equiv [0, L_1]$ together with their appropriate derivatives. Fluctuation amplitudes are governed by the system of linear algebraic equations $(5.15)_{3,4}$ and can be always eliminated from the system of governing equations (5.15) by means of (5.16). Hence, ***the unknowns of final asymptotic model equations*** (5.18) are only macrodisplacements. The resulting equations have to be considered together with decomposition (5.19). ***Coefficients in the asymptotic equations are continuously slowly variable in x, but they are independent of the microstructure cell size. Thus, contrary to the tolerance model, the consistent asymptotic one is not able to describe the length-scale effect on the overall shell dynamics. The great advantage of this model is that the effective moduli (5.17) of the shell can be obtained without specification of the periodic cell problems.***

c) ***The new mathematical combined asymptotic-tolerance model*** for the analysis of selected dynamic problems in the functionally graded shells under consideration ***has been formulated by applying the combined modelling procedure*** given in Woźniak *et al.* (eds.) [164]. The combined modelling technique is realized in two steps. The first step is based on ***the consistent asymptotic procedure*** (outlined above), which leads from starting equations (4.10) to the Euler-Lagrange equations (5.18) with continuous and slowly-varying *coefficients* being independent of the microstructure cell size. Equations (5.18) without the external forces are rewritten as (5.20). Equations (5.20) are referred to as ***the macroscopic model*** equations. Assuming that in the framework of the macroscopic model the solutions (5.21) to the problem under consideration are known, we can pass to the second step. The second step is based on ***the tolerance averaging of starting lagrangian (4.8) under so-called superimposed decomposition defined by (5.22)***. This *extra micro-macro decomposition* superimposed on the known solutions obtained within the macroscopic model contains ***the new known, tolerance-periodic, continuous and highly oscillating fluctuation shape functions and new slowly-varying unknowns termed fluctuation amplitudes.*** Then, applying the principle of stationary action to the tolerance-averaged action functional (5.28) defined by means of the tolerance-averaged lagrangian (5.27) we arrive at the Euler-Lagrange equations (5.29) and their explicit form (5.30), (5.31) with *continuous and slowly-varying in x coefficients depending also on the cell size.* Hence, the model obtained in the second step is referred to as ***the superimposed microscopic model.*** Summarizing the results we conclude that ***the equations of combined model for the tolerance-periodic shells under consideration consist of macroscopic model equations (5.20) formulated by means of the consistent asymptotic procedure and having continuous and slowly changing coefficients independent of a microstructure length and of superimposed microscopic model equations (5.30), (5.31) derived by applying the tolerance modelling technique and having continuous and slowly-varying coefficients depending also on a cell size. Both the models are combined together under assumption that in the framework of the macroscopic model the solutions (5.21) to the problem under consideration are known.*** It was shown *that under special condition imposed on the fluctuation shape functions*, ***the combined model makes it possible to separate the macroscopic description of some special problems from their microscopic description***; see equations (5.34)-(5.36). Thus, ***an important advantage of this model is that it allows us to study micro-dynamics of the shells under consideration independently of their macro-dynamics.*** Moreover, micro-dynamic equations (5.34)-(5.36) describe certain *time-boundary-layer and space-boundary-layer phenomena* strictly related to the specific form of initial and boundary conditions imposed on the unknown micro-fluctuation amplitudes.

d) The governing equations of all models derived in this dissertation are uniquely determined by the continuous, highly oscillating, tolerance-periodic *fluctuations shape functions* describing oscillations inside a cell. These functions are assumed to be known in every problem under consideration. They can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell. It means that they represent either the principal modes of free periodic vibrations of the cell or physically reasonable approximation of these modes. These functions can also be treated as the shape functions resulting from the periodic discretization of the cell using for example the finite element method. The choice of these functions can be also based on the experience or intuition of the researcher.

e) The number and form of boundary conditions for unknown averaged variables (i.e. macrodisplacements) of all models formulated here are the same as in the classical shell theory governed by equations (4.10). Boundary conditions for unknown fluctuation amplitudes should be defined only on boundaries $\xi = 0$, $\xi = L_2$.

f) Solutions to selected initial/boundary value problems formulated in the framework of the tolerance model and the microscopic part of combined model have a physical sense only if the physical reliability conditions hold for the pertinent tolerance parameters. These conditions state that unknowns of the aforementioned models have to be slowly-varying functions in direction of tolerance-periodicity. Moreover, these conditions can be also used for the *a posteriori* evaluation of tolerance parameters and hence, for the verification of the physical reliability of the obtained solutions.

The second aim of this doctoral thesis was to apply the tolerance and asymptotic models derived here to evaluation of the length-scale effect in some special problems dealing with free vibrations of the tolerance-periodic shells under consideration. The objects of considerations were the simply supported shell strip of infinite axial length dimension and the open simply supported shell of finite circumferential and axial length dimensions.

The shells have constant thickness and are composed of two homogeneous, elastic, isotropic materials densely and tolerance-periodically distributed along the circumferential direction. The materials are perfectly bonded on interfaces. The shell structure is constant in the axial direction, cf. Fig. 6.1. On the macroscopic level the shells have transversally graded macrostructure in circumferential direction.

Many functions describing the distribution of material properties have been taken into account, cf. expressions (6.6)-(6.16) and Fig. 6.3.

The analysed free vibration problems were described by equations with coefficients continuous and slowly-varying in x. It was difficult to find analytical solutions to these equations. Thus, to obtain approximate formulas of free vibration frequencies, the known Ritz variational method was applied.

Some analytical results derived in the framework of the tolerance and asymptotic models were compared with numerical those obtained using the commercial computer software Ansys based on the finite element methods.

The most important conclusions are:

a) **Contrary to asymptotic model, the tolerance one describes the effect of a cell size on the dynamic behaviour of the functionally graded shells under consideration.** ***In the framework of the tolerance model, not only the fundamental lower, but also the <u>new additional higher-order free vibration frequencies</u> can be derived and analysed. The higher free vibration frequencies depend on a cell size and hence cannot be determined applying asymptotic models commonly used for investigations of the microstructured shell dynamics.*** In the special dynamic problems discussed here, ***the cell-dependent higher-order free vibration frequencies*** are expressed by means of formulae (6.29) for a shell strip and (6.53) for the shell with finite length dimensions.

b) From both the analytical and computational results it follows that the differences between the fundamental lower free vibration frequencies derived from the tolerance model and free vibration frequencies obtained from the asymptotic one are ***negligibly small.*** Thus, *the effect of the microstructure size on the fundamental lower free vibration frequencies of the shells under consideration can be neglected.* Hence, ***the asymptotic model being more simple than the tolerance non-asymptotic one is sufficient from the point of view of calculations made for the dynamic problems under consideration.***

c) Values of the free vibration frequencies derived from the tolerance or asymptotic models increase with decreasing differences between elastic properties of the shell component materials and decrease with decreasing differences between inertial properties of the shell component materials. Values of these frequencies increase linearly with decreasing of differences between shell thickness and the microstructure length parameter λ. Values of the cell-dependent higher free vibration frequencies decrease with the decreasing of differences between a cell size λ and the length dimension $L \equiv L_1$ of the shell midsurface in the tolerant periodicity direction. For every distribution function under consideration and for $\lambda/L \in [0.01, 0.1]$, these values decrease very strongly for $\lambda/L \in [0.01, 0.03]$.

d) The comparison of the lower free vibration frequencies derived from the tolerance or asymptotic models with those obtained using the commercial computer software Ansys has given a very good agreement between these results. Good agreement confirms the validation of the models proposed here.

The third aim was to apply the microscopic equations (5.34)-(5.36) derived in the second step of the combined modelling to the analysis of length-scale effect in some special problems dealing with the shell micro-vibrations and with the long wave propagation related to micro-fluctuations of the shell displacements. The open shell having constant thickness and composed of two homogeneous, elastic, isotropic materials densely and tolerance-periodically distributed along the circumferential direction is object of considerations, cf. Fig. 6.1. The component materials are perfectly bonded on interfaces. The shell structure is constant in an axial direction. On the macroscopic level the shells have transversally graded macrostructure in circumferential direction.

It was shown that the combined model for the tolerance-periodic shells considered here, under special conditions imposed on the fluctuation shape functions, makes it possible to analyse selected problems of the shell micro-dynamics independently of the shell macro-dynamics. This is the greatest advantage of the proposed combined model. Note, that the problems mentioned above cannot be analysed in the framework of the asymptotic models commonly used for investigations of dynamics of periodic/tolerance-periodic cylindrical shells. Micro-dynamic equations (5.34)-(5.36) obtained in the second step of the combined modelling are independent of solutions obtained in the framework of the macroscopic (i.e. asymptotic) model formulated in the first step of the combined modelling. We recall that coefficients of micro-dynamic equations (5.34)-(5.36) are described by continuous and slowly-varying functions with respect to argument x.

The results obtained on the basis of micro-dynamic equations (5.34)-(5.36) can be summarized by the following remarks and conclusions:

a) For the transversally graded shell under consideration simply supported on all four edges, ***the cell-dependent free micro-vibration frequencies*** (7.4), (7.8), (7.12) in circumferential, axial and normal directions, respectively, have been derived on the basis of micro-dynamic equations (5.34)-(5.36) and by applying the known approximate Galerkin method. These frequencies were obtained independently of ***the cell-independent free macro-vibration frequencies.*** The dependence of these frequencies on the microstructure length parameter λ was studied. The influence of differences between elastic and inertial properties of the component materials on these frequencies was investigated in detail. Linear, parabolic, sinus and constant (i.e. periodic) distributions of material properties were taken into account. The highest values of the free micro-vibration frequencies in circumferential and axial directions were obtained for the sinus distribution and for a shell with elastic homogeneous structure and a very strong inertial heterogeneity, while the lowest values were obtained for parabolic distribution and for a shell having inertial homogeneous structure and a very strong elastic inhomogeneity. The highest value of the transversal free micro-vibration frequency was obtained for the parabolic distribution and for an elastically homogeneous shell with a very strong inertial heterogeneity. The lowest value of this frequency was also obtained for the parabolic distribution, but for a shell with inertial homogeneity and a very strong elastic heterogeneity. The free micro-vibration frequencies derived in the framework of the combined model can be also obtained applying the tolerance one (5.6), (5.7). However, tolerance model equations (5.6), (5.7) are much more complicated that the microscopic equations (5.34)-(5.36) of the combined model. Moreover, within the tolerance model the *cell-dependent higher free vibration frequencies* are always determined not separately but together with the fundamental *cell-independent lower free vibration frequencies*. In order to check this conformability, the transversal free micro-vibration frequencies (7.12) obtained on the basis of micro-dynamic equations (5.36) of the combined model were compared with corresponding those (6.53) derived in the framework of tolerance model (5.6), (5.7). Obviously, calculations based on (6.53) were carried out for $m = n = 1$, i.e. for wave numbers $\alpha = \pi/L_1$, $\beta = \pi/L_2$. It has been shown that the results from the combined model are in a very good agreement with those from the tolerance model.

b) The length-scale effect in a special problem for the open functionally graded shell under consideration dealing with the axial harmonic micro-vibrations with vibration frequency $\breve{\omega}$ was analysed on the basis of micro-dynamic equation (5.35), which in the studied problem reduces to Eq. (7.16). Special boundary conditions were imposed on fluctuation amplitude being unknown in this equation. Linear, parabolic, sinus and constant (i.e. periodic) distributions of material properties were taken into account. It was shown that the shape of these micro-vibrations depends on relations between values of frequency $\breve{\omega}$ and ***a certain new additional higher-order free vibration frequency $\breve{\omega}_*$ (7.19)$_2$ depending on a cell size λ. The micro-vibrations can decay exponentially and very strongly near the boundary $\xi \equiv x^2 = 0$ and can be treated as equal to zero outside a certain narrow layer near this boundary***. The problem with strongly decaying micro-vibrations near the boundary $\xi = 0$ is referred to as ***the space-boundary layer phenomena. Thus, it was shown that the microscopic equations of the combined model describe the space-boundary layer phenomena. The micro-vibrations can decay exponentially but not so strongly. They can decay linearly. Certain values of $\breve{\omega}$ cause a non-decayed form of micro-vibrations (micro-vibrations oscillate), for certain values of $\breve{\omega}$ we deal with resonance micro-vibrations.***

c) The problem of long wave propagation in the open functionally graded shell under consideration but now unbounded in an axial direction was analysed on the basis of equation (5.35) describing the shell micro-dynamics in an axial direction. In the considered problem, this equation reduces to Eq. (7.16). Linear, parabolic, cosine and constant (i.e. periodic) distributions of material properties were taken into account. The long waves, related to micro-fluctuation amplitude being unknown of equation (7.16) were studied. Note, that we deal with long waves if condition $\lambda/\overline{L} \ll 1$ holds, where λ is the characteristic length dimension of a cell and $\overline{L}$ is the wavelength. It was shown that the ***tolerance-periodically micro-heterogeneity of the shell leads to exponential waves and to dispersion effects, which cannot be analysed in the framework of the asymptotic models for periodic/tolerance-periodic shells.*** Moreover, ***the new wave propagation speed (7.35) depending on the microstructure size has been obtained.*** The influence of the shell elastic and inertial properties on this cell-dependent speed was analysed. From numerical calculations, it follows that the values of the wave propagation velocity increase with decreasing differences between elastic properties of the shell component materials, but they decrease with decreasing of differences between inertial properties of the component materials. Values of the wave propagation speed decrease with the decreasing of differences between a cell size λ and the wavelength $\overline{L}$.

d) The influence of a microstructure size on the character of displacement micro-fluctuations in the transversally graded shell under consideration was also analysed in a certain special initial-value problem describing by equation (7.16) (or its dimensionless form (7.17)). We recall that Eq. (7.16) describes micro-dynamic behaviour of the shell in an axial direction. The considerations were restricted to the time interval $[0, \pi/2]$. The important conclusion is that the distribution of these micro-fluctuations in the time interval under consideration depends on interrelations between microstructure length parameter λ and a certain length parameter l independent of a cell size and described by means of elastic and inertial properties of the shell and by a certain time constant. It has been shown that if $l < \lambda$, then the displacement micro-fluctuations decrease monotonically and very softly; they don't take the zero-value in the time interval under consideration. For $l = \lambda$ the micro-fluctuations decay monotonically in the time interval under consideration; for $\tau = \pi/2$ they are equal to zero. If $l > \lambda$ then micro-fluctuations decay monotonically and strongly in a certain subinterval of the time interval under consideration, and then the absolute values of these solutions increase monotonically in the remaining part of this time interval.

We recall that some special engineering problems discussed in Chapters 7 and 8 are related to shells composed of two homogeneous, elastic, isotropic materials densely and tolerance-periodically distributed along the circumferential direction. It has to be emphasized that the results presented in the chapters mentioned above can be easily extended on the case in which we deal with shells composed of more than two homogeneous, elastic, isotropic materials densely and tolerance-periodically distributed along x-coordinate. It is worth mentioning that the averaged models proposed in the doctoral thesis can also be applied to investigations of dynamic problems in cylindrical shells reinforced by stiffeners tolerance-periodically and densely distributed in circumferential direction. On the macroscopic level, these stiffened shells can be treated as shells with transversally graded macrostructure.

The functionally graded shells being objects of considerations in this doctoral dissertation are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as elements of housings of reactors and tanks. The transversally graded shells having small length dimensions are elements of air-planes, ships and machines.

The most important original elements of the doctoral thesis are as follows:

1. Derivation of three new mathematical averaged models for the analysis of dynamic problems in thin linearly elastic Kirchhoff-Love-type open

circular cylindrical shells having a functionally graded macrostructure and a tolerance-periodic microstructure in the circumferential direction as well as with a constant structure in an axial direction:

a) the tolerance non-asymptotic model ***governed by equations with continuous and slowly-varying coefficients depending also on a microstructure size***, cf. (5.6), (5.7); it means this model makes it possible to investigate ***the length-scale effect***,

b) the consistent asymptotic model ***governed by equations with continuous and slowly-varying coefficients but independent of a cell size***, cf. (5.18),

c) the combined asymptotic-tolerance model in which asymptotic and tolerance models are conjugated with themselves under assumption that solutions obtained in the framework of the asymptotic model are known, cf. (5.18), (5.30), (5.31); ***coefficients of this model are continuous and slowly varying; moreover, coefficients in microscopic (tolerance) model (5.30), (5.31) which is imposed on the macroscopic (asymptotic) one (5.18) depend on microstructure size.***

2. It has been shown that the tolerance model formulated here can be successfully applied to investigations of the effect of a cell size on the dynamic behaviour of the functionally graded cylindrical shells under consideration. It makes it possible to determine and analyse ***the new additional cell-dependent higher free vibration frequencies caused by the tolerance-periodic structure of the shells.***

3. It has been given evidence that the differences between the fundamental lower free vibration frequencies derived from the tolerance model and free vibration frequencies obtained from the asymptotic one are negligibly small. Hence, from a calculation point of view, the asymptotic model being more simple than the non-asymptotic one is sufficient for the determination and analysis of the basic free vibration frequencies.

4. It has been shown that the microscopic tolerance equations derived in the second step of combined modelling, under special conditions imposed on the fluctuation shape functions, are independent of solutions obtained in the first step of combined modelling, i.e. in the framework of asymptotic model., cf. (5.34)-(5.36). ***It means that microscopic equations (5.34)-(5.36) make it possible to study the shell micro-dynamics independently of the shell macro-dynamics. This is the greatest advantage of the combined model proposed in this dissertation.***

a) Using these micro-dynamic equations, ***the new cell-dependent higher free vibration frequencies have been determined and analysed independently of the fundamental, classical cell-independent lower free vibration frequencies.***

b) ***Some new important results have been obtained analysing the harmonic micro-vibrations*** with vibration frequency $\breve{\omega}$. It was shown that the form of these micro-vibrations depends on relations between values of vibration frequency $\breve{\omega}$ and a certain new additional higher-order free vibration frequency $\breve{\omega}_*$ $(7.19)_2$ depending on the cell size. ***The micro-vibrations can decay exponentially. They can decay linearly. For certain interrelations between $\breve{\omega}$ and $\breve{\omega}_*$ we deal with a non-decayed form of micro-vibrations (micro-vibrations oscillate) or with resonance micro-vibrations. Moreover, it was shown that the micro-dynamic equations of the combined model describe the space-boundary layer phenomena.***

c) ***Some new important results have been obtained analysing the long wave propagation problem related to micro-fluctuations in axial direction***. It was shown that the tolerance-periodic micro-heterogeneity of the shells leads to exponential waves and to dispersion effects. Moreover, the new wave propagation speed (7.35) depending on the microstructure size has been obtained.

d) ***Some new important results have been obtained*** examining the influence of a microstructure size on the character of displacement micro-fluctuations in a certain initial-value problem with special initial conditions. It has been given evidence that the distribution of these micro-fluctuations in the time interval under consideration depends on interrelations between microstructure length parameter λ and a certain length parameter l independent of a cell size and described by means of elastic and inertial properties of the shell and by a certain time constant.

The most important final comments are:

1. ***The assumed aims of this doctoral thesis have been realized.***

2. ***Theses of the doctoral dissertation have been proven.***

The results obtained in this dissertation have an essential influence on the state of knowledge dealing with dynamic behaviour of thin-walled tolerance-periodic cylindrical shells, which on the macro-level are referred to as the transversally graded shells. The results also generate

new directions of further investigations. Thus, the results exert an influence on the development of this field of knowledge.

The anticipated directions of further investigations can be related to:

- the modelling of dynamic and stability problems for the cylindrical shells of a heterogeneous microstructure, which is periodic in the circumferential direction and slowly varying along the axial direction (***longitudinally graded shells***),
- the stationary and dynamic stability problems,
- the analysis of shells in the framework of theories which are more exact than the Kirchhoff-Love shell theory,
- the geometrically non-linear shell problems,
- the modelling of dynamic thermoelasticity problems and others.

Appendix: Calculations of coefficients in averaged models equations

Averaged models equations discussed in Chapters 6 and 7 have coefficients shown below.

Note, that in calculations of those coefficients, the dimensional forms of functions $h(\cdot) \in O(\lambda)$, $g(\cdot) \in O(\lambda^2)$, $\partial_1 g(\cdot) \in O(\lambda)$ are taken into account, cf. (A.5), (A.7), (A.14)-(A.18), (A.20), (A.21). Some of the averages found in the equations discussed in Chapters 6, 7 contain dimensionless forms of these functions, i.e. $\overline{h} \equiv \lambda^{-1}h$, $\overline{g} \equiv \lambda^{-2}g$, $\widehat{g} \equiv \lambda^{-1}g$.

$$\begin{aligned}
&\left\langle D^{1111}\right\rangle = \left\langle D^{2222}\right\rangle = \\
&= \frac{d}{\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{dE_1}{1-\nu^2}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x) + \frac{E_2}{E_1}\right]
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\left\langle D^{1122}\right\rangle = \frac{d\nu}{\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{d\nu E_1}{1-\nu^2}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x) + \frac{E_2}{E_1}\right]
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\left\langle D^{1212}\right\rangle = \frac{d}{2\lambda\left(1+\nu\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{dE_1}{2(1+\nu)}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x) + \frac{E_2}{E_1}\right]
\end{aligned} \tag{A.3}$$

$$\left\langle D^{1111}\left(\partial_1 h\right)^2\right\rangle = \frac{d}{\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz + \right.$$
$$\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz\right) = \tag{A.4}$$
$$= \frac{2\pi d E_1}{1-\nu^2}\left[\left(1-\frac{E_2}{E_1}\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)+\pi\widetilde{\eta}(x)\right)+\frac{E_2}{E_1}\pi\right]$$

$$\left\langle D^{1212} h^2\right\rangle = \frac{(1-\nu)\, d}{2\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\widetilde{h}(\cdot,z)\right)^2 dz + \right.$$
$$\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\widetilde{h}(\cdot,z)\right)^2 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\widetilde{h}(\cdot,z)\right)^2 dz\right) = \tag{A.5}$$
$$= \frac{\lambda^2 d E_1}{4\pi\left(\nu+1\right)}\left[\left(\frac{E_2}{E_1}-1\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)-\pi\widetilde{\eta}(x)\right)+\frac{E_2}{E_1}\pi\right]$$

$$\left\langle D^{1212}\left(\partial_1 h\right)^2\right\rangle = \frac{(1-\nu)\, d}{2\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz + \right.$$
$$\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\partial_1\widetilde{h}(\cdot,z)\right)^2 dz\right) = \tag{A.6}$$
$$= \frac{\pi d E_1}{\nu+1}\left[\left(1-\frac{E_2}{E_1}\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)+\pi\widetilde{\eta}(x)\right)+\frac{E_2}{E_1}\pi\right]$$

$$
\begin{aligned}
\left\langle D^{2222}h^2\right\rangle &= \frac{d}{\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\widetilde{h}(\cdot,z)\right)^2 dz + \right.\\
&\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\widetilde{h}(\cdot,z)\right)^2 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\widetilde{h}(\cdot,z)\right)^2 dz\right) = \\
&= \frac{\lambda^2 d E_1}{2\pi\left(1-\nu^2\right)}\left[\left(\frac{E_2}{E_1}-1\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)-\pi\widetilde{\eta}(x)\right)+\frac{E_2}{E_1}\pi\right]
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
\left\langle B^{1111}\right\rangle &= \left\langle B^{2222}\right\rangle = \\
&= \frac{d^3}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{d^3 E_1}{12\left(1-\nu^2\right)}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x)+\frac{E_2}{E_1}\right]
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\left\langle B^{1122}\right\rangle &= \frac{d^3\nu}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{d^3\nu E_1}{12\left(1-\nu^2\right)}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x)+\frac{E_2}{E_1}\right]
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\left\langle B^{1212}\right\rangle &= \frac{d^3}{24\lambda\left(1+\nu\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 dz + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 dz\right) = \\
&= \frac{d^3 E_1}{24\left(1+\nu\right)}\left[\left(1-\frac{E_2}{E_1}\right)\widetilde{\eta}(x)+\frac{E_2}{E_1}\right]
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\left\langle B^{1111}\partial_{11}g\right\rangle &= \frac{d^3}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z+\right.\\
&\left.+\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z\right)=\\
&=\frac{\pi d^3 E_1}{3\left(1-\nu^2\right)}\left[\left(\frac{E_2}{E_1}-1\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\right)\right]
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
\left\langle B^{1122}\partial_{11}g\right\rangle &= \frac{d^3\nu}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z+\right.\\
&\left.+\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\partial_{11}\widetilde{g}(\cdot,z)\mathrm{d}z\right)=\\
&=\frac{\pi d^3 \nu E_1}{3\left(1-\nu^2\right)}\left[\left(\frac{E_2}{E_1}-1\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\right)\right]
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
\left\langle B^{1111}\left(\partial_{11}g\right)^2\right\rangle &= \frac{d^3}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\partial_{11}\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z+\right.\\
&\left.+\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\partial_{11}\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\partial_{11}\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z\right)=\\
&=\frac{2\pi^3 d^3 E_1}{3\left(1-\nu^2\right)}\left[\left(1-\frac{E_2}{E_1}\right)\left(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)+\pi\widetilde{\eta}(x)\right)+\frac{E_2}{E_1}\pi\right]
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
\left\langle B^{2222}g\right\rangle &= \frac{d^3}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\widetilde{g}(\cdot,z)\mathrm{d}z+\right.\\
&+\left.\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\widetilde{g}(\cdot,z)\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\widetilde{g}(\cdot,z)2\mathrm{d}z\right)=\\
&=\frac{\lambda^2d^3E_1}{12\pi\left(1-\nu^2\right)}\left[\left(\frac{E_2}{E_1}-1\right)\Big(\sin\left(\pi\widetilde{\eta}(x)\right)+\pi c(x)\widetilde{\eta}(x)\Big)-\pi c(x)\right]
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
\left\langle B^{1122}g\right\rangle &= \frac{d^3\nu}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\widetilde{g}(\cdot,z)\mathrm{d}z+\right.\\
&+\left.\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\widetilde{g}(\cdot,z)\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\widetilde{g}(\cdot,z)2\mathrm{d}z\right)=\\
&=\frac{\lambda^2d^3\nu E_1}{12\pi\left(1-\nu^2\right)}\left[\left(\frac{E_2}{E_1}-1\right)\Big(\sin\left(\pi\widetilde{\eta}(x)\right)+\pi c(x)\widetilde{\eta}(x)\Big)-\pi c(x)\right]
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
\left\langle B^{2222}g^2\right\rangle &= \frac{d^3}{12\lambda\left(1-\nu^2\right)}\left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2\left(\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z+\right.\\
&+\left.\int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1\left(\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z+\int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2\left(\widetilde{g}(\cdot,z)\right)^2\mathrm{d}z\right)=\\
&=\frac{\lambda^4d^3E_1}{24\pi\left(1-\nu^2\right)}\left[\left(1-\frac{E_2}{E_1}\right)\Big(\sin\left(\pi\widetilde{\eta}(x)\right)\cos\left(\pi\widetilde{\eta}(x)\right)+\right.\\
&\left.+4c(x)\sin\left(\pi\widetilde{\eta}(x)\right)+2\pi\widetilde{\eta}(x)c(x)^2+\pi\widetilde{\eta}(x)\Big)+\frac{E_2}{E_1}\left(2\pi c(x)^2+\pi\right)\right]
\end{aligned}
\tag{A.16}
$$

$$\left\langle B^{1122} g \partial_{11} g \right\rangle = \frac{\nu d^3}{12\lambda\left(1-\nu^2\right)} \left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 \left(\partial_{11}\widetilde{g}(\cdot,z)\right) \widetilde{g}(\cdot,z) \mathrm{d}z + \right.$$

$$\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 \left(\partial_{11}\widetilde{g}(\cdot,z)\right) \widetilde{g}(\cdot,z) \mathrm{d}z + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 \left(\partial_{11}\widetilde{g}(\cdot,z)\right) \widetilde{g}(\cdot,z) \mathrm{d}z \right) = \tag{A.17}$$

$$= \frac{\lambda^2 \pi d^3 \nu E_1}{6\left(1-\nu^2\right)} \left[\left(\frac{E_2}{E_1} - 1\right) \Big(\sin\left(\pi\widetilde{\eta}(x)\right) \cos\left(\pi\widetilde{\eta}(x)\right) + \right.$$

$$\left. + 2c(x) \sin\left(\pi\widetilde{\eta}(x)\right) + \pi\widetilde{\eta}(x) \Big) - \frac{E_2}{E_1}\pi \right]$$

$$\left\langle B^{1212} \left(\partial_1 g\right)^2 \right\rangle = \frac{(1-\nu)\, d^3}{24\lambda\left(1-\nu^2\right)} \left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} E_2 \left(\partial_1\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z + \right.$$

$$\left. + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} E_1 \left(\partial_1\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} E_2 \left(\partial_1\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z \right) = \tag{A.18}$$

$$= \frac{\lambda^2 \pi d^3 \nu E_1}{12\left(\nu+1\right)} \left[\left(\frac{E_2}{E_1} - 1\right) \Big(\sin\left(\pi\widetilde{\eta}(x)\right) \cos\left(\pi\widetilde{\eta}(x)\right) - \pi\widetilde{\eta}(x) \Big) + \frac{E_2}{E_1}\pi \right]$$

$$\langle \mu \rangle = \frac{d}{\lambda} \left(\int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_2 \mathrm{d}z + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_1 \mathrm{d}z + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} \rho_2 \mathrm{d}z \right) = \tag{A.19}$$

$$= d\rho_1 \left[\widetilde{\eta}(x) \left(1 - \frac{\rho_2}{\rho_1}\right) + \frac{\rho_2}{\rho_1} \right]$$

$$
\begin{aligned}
\left\langle \mu h^2 \right\rangle = \frac{d}{\lambda} \Bigg(& \int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_2 \left(\widetilde{h}(\cdot,z)\right)^2 dz + \\
& + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_1 \left(\widetilde{h}(\cdot,z)\right)^2 dz + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} \rho_2 \left(\widetilde{h}(\cdot,z)\right)^2 dz \Bigg) = \\
& = \frac{\lambda^2 d\rho_1}{2\pi} \left[\left(\frac{\rho_2}{\rho_1} - 1\right) \left(\sin\left(\pi\widetilde{\eta}(x)\right) \cos\left(\pi\widetilde{\eta}(x)\right) - \pi\widetilde{\eta}(x) \right) + \frac{\rho_2}{\rho_1}\pi \right]
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
\left\langle \mu g^2 \right\rangle = \frac{d}{\lambda} \Bigg(& \int\limits_{-\frac{\lambda}{2}}^{-\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_2 \left(\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z + \\
& + \int\limits_{-\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{1}{2}\widetilde{\eta}(x)\lambda} \rho_1 \left(\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z + \int\limits_{\frac{1}{2}\widetilde{\eta}(x)\lambda}^{\frac{\lambda}{2}} \rho_2 \left(\widetilde{g}(\cdot,z)\right)^2 \mathrm{d}z \Bigg) = \\
& = \frac{\lambda^4 d\rho_1}{2\pi} \Bigg[\left(1 - \frac{\rho_2}{\rho_1}\right) \Big(\sin\left(\pi\widetilde{\eta}(x)\right) \cos\left(\pi\widetilde{\eta}(x)\right) + \\
& + 4c(x)\sin\left(\pi\widetilde{\eta}(x)\right) + 2\pi\widetilde{\eta}(x)c(x)^2 + \pi\widetilde{\eta} \Big) + \frac{\rho_2}{\rho_1}\left(2\pi c(x)^2 + \pi\right) \Bigg]
\end{aligned}
\tag{A.21}
$$

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Summary

Dynamics of thin functionally graded cylindrical shells - tolerance modelling

The subject-matter of this doctoral thesis is the analytical modelling and analysis of dynamic problems for thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells having ***a functionally graded macrostructure*** and ***a tolerance-periodic microstructure*** in circumferential direction. It means that ***on the microscopic level***, the shells consist of a very large number of separated, small elements regularly distributed along circumferential direction and perfectly bonded to each other (or to the homogeneous matrix), cf. Figs. 4.1 and 4.2. These elements, called ***cells***, are treated as thin shells. It is assumed that ***the adjacent cells are nearly identical*** (*i.e. they have nearly the same geometrical, elastic and inertial properties*), ***but the distant elements can be very different.*** At the same time, the shells have constant structure in axial direction. ***On the microscopic level***, the geometrical, elastic and inertial properties of these shells are determined by highly oscillating, non-continuous and *tolerance-periodic functions* in circumferential direction. On the other hand, ***on the macroscopic level***, the averaged properties of the shells are described by functions being ***continuous and slowly varying*** along the direction of tolerant periodicity. It means that the tolerance-periodic shells under consideration can be treated as made of ***functionally graded materials*** (FGM) and called ***functionally graded shells.*** Moreover, since macroscopic (i.e. averaged) properties of the shells are graded in direction normal to interfaces between constituents, this gradation is referred to as ***the transversal gradation.***

Dynamic behaviour of such shells are described by the known governing equations (4.10) of Kirchhoff-Love theory for thin elastic shells. For tolerance-periodic shells, coefficients of these equations are highly oscillating, non-continuous and tolerance-periodic functions. That is why the direct application of these equations to investigations of specific dynamic problems is non-effective even using computational methods.

The first aim of the doctoral thesis has been to formulate and

discuss three new mathematical averaged models with continuously slowly-varying coefficients constituting a proper tool for the analysis of selected dynamic problems in the thin cylindrical shells with a tolerance-periodic microstructure and transversally graded macrostructure in the circumferential direction. Moreover, two from these models take into account the effect of a microstructure size on the dynamic shell behaviour. In order to formulate these models, the tolerance, consistent asymptotic and combined asymptotic-tolerance modelling procedures, cf. [164], have been applied to the starting Euler-Lagrange equations (4.9), which explicit form (4.10) coincides with the governing equations of Kirchhoff-Love theory for thin elastic shells.

The ***tolerance approach*** is based on the notion of ***tolerance relations*** between points and real numbers related to the accuracy of the performed measurements and calculations. Tolerance relations are determined by ***tolerance parameters.*** Other fundamental concepts of this modelling technique are those of ***slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and averaging operation.*** The tolerance modelling is based on two assumptions. The first of them is called ***the tolerance averaging approximation*** and makes it possible to neglect terms of an order of tolerance parameters. The second one is termed ***the micro-macro decomposition.*** It states that the displacement fields occurring in the starting lagrangian have to be *the tolerance-periodic functions* in the direction of tolerant periodicity. Hence, they can be decomposed into ***unknown averaged displacements (macrodisplacements) being slowly-varying functions and fluctuations represented by finite series of products of the known highly oscillating continuous tolerance-periodic fluctuation shape functions and unknown slowly-varying fluctuation amplitudes.*** The basic concepts and assumptions of the tolerance modelling technique are outlined in Chapter 3 of this doctoral thesis.

The tolerance modelling technique applied to starting Euler-Lagrange equations (4.9) has been realized in two steps. The first step has been based on ***the tolerance averaging of the starting lagrangian (4.8)*** by applying ***micro-macro decomposition (5.1)***, ***averaging operation (3.5)*** as well as ***the tolerance averaging approximation (3.6)***. The resulting averaged form of lagrangian (4.8) is given by (5.3). In the second step, using *the principle of stationary action* to ***the averaged action functional (5.4)*** defined by means of ***tolerance-averaged lagrangian (5.3)***, we have arrived at Euler-Lagrange equations (5.5) and then at their explicit form given by constitutive relations (5.6) and dynamic balance equations (5.7). Summarizing, ***the***

tolerance model for the analysis of dynamic problems for thin linearly elastic Kirchhoff-Love-type cylindrical shells having a transversally graded macrostructure and a tolerance-periodic microstructure in circumferential direction is represented by constitutive relations (5.6) and dynamic balance equations (5.7) together with micro-macro decomposition (5.1) and physical reliability conditions (5.2). The basic unknowns are ***macrodisplacements*** and ***fluctuation amplitudes*** which must be slowly-varying along x-coordinate parametrizing the shell midsurface in circumferential direction. ***The resulting tolerance model equations have coefficients which are continuous and slowly-varying in the direction of tolerant periodicity. Moreover, some of these coefficients depend on microstructure size. The length-scale effect can be analysed not only in dynamic but also in stationary problems.***

On passing from tolerance averaging to the consistent asymptotic averaging, the concept of highly oscillating fluctuation shape functions is retained only. The notions of tolerance-periodic functions and slowly-varying functions are not introduced. The fundamental assumption imposed on the starting lagrangian in the framework of this approach is called ***the consistent asymptotic decomposition.*** It states that the displacement fields occurring in the lagrangian have to be replaced by families of fields depending on parameter $\varepsilon \in (0,1]$ and defined in an arbitrary cell. These families of displacements are decomposed into averaged part described by unknown functions (macrodisplacements) being continuously bounded in the tolerant periodicity direction and highly-oscillating part depending on ε. This highly-oscillating part is represented by the known highly oscillating fluctuation shape functions multiplied by unknown functions (fluctuation amplitudes) being continuously bounded in the direction of tolerant periodicity.

Asymptotic modelling procedure applied to Euler-Lagrange equations (4.9) has been realized in two steps. The first step has been ***the consistent asymptotic averaging of lagrangian (4.9) under consistent asymptotic decomposition defined by (5.8).*** The resulting averaged form of lagrangian (4.9) is given by (5.12). In the second step, applying *the principle of stationary action* to *the consistent asymptotic action functional* (5.13) defined by means of averaged lagrangian (5.12), we have arrived at Euler-Lagrange equations (5.14) and then at their explicit form (5.15). Finally, after eliminating unknown fluctuation amplitudes by means of (5.16), we have obtained ***asymptotic model equations*** (5.18) expressed only in ***macrodisplacements.*** The resulting equations have to be considered together with decomposition (5.19). ***Coefficients in the asymptotic equations are continuously slowly variable in x, but they are independent of the microstructure cell size.*** Thus, ***contrary to the tolerance model, the***

consistent asymptotic one is not able to describe the length-scale effect on the overall shell dynamics.

The combined asymptotic-tolerance model for the analysis of selected dynamic problems in the functionally graded shells under consideration ***has been formulated by applying the combined modelling procedure*** given in Woźniak *et al.* (eds.) [164]. This combined modelling includes the consistent asymptotic and the tolerance non-asymptotic modelling techniques which are combined together into a new procedure. ***The equations of combined model proposed here consist of asymptotic (macroscopic) model equations (5.20) formulated by means of the consistent asymptotic procedure and having continuous and slowly changing coefficients independent of a microstructure length and of superimposed tolerance (microscopic) model equations (5.30), (5.31) derived by applying the tolerance modelling technique and having continuous and slowly-varying coefficients depending also on a cell size. Both the models are combined together under assumption that in the framework of the asymptotic model the solutions to the problem under consideration are known***. It has been shown *that under special condition imposed on the fluctuation shape functions*, ***the combined model makes it possible to separate the macroscopic description of some special problems from their microscopic description***, cf. equations (5.34)-(5.36). Thus, ***an important advantage of this model is that it allows us to study micro-dynamics of the shells under consideration independently of their macro-dynamics.***

Solutions to selected initial/boundary value problems formulated in the framework of the tolerance model and the microscopic part of combined model have a physical sense only if unknowns of the aforementioned models are slowly-varying functions in the direction of tolerant periodicity. Moreover, these conditions can be also used for the *a posterior*i evaluation of tolerance parameters and hence, for the verification of the physical reliability of the obtained solutions.

The second aim of this doctoral thesis has been to apply the tolerance and asymptotic models derived here to evaluation of the length-scale effect in some special problems dealing with free vibrations of the tolerance-periodic shells under consideration. In order to find analytical solutions to the governing equations of these models (equations with continuous and slowly-varying coefficients), the known Ritz variational method has been applied. It has been shown that ***in the framework of the tolerance model, not only the fundamental cell-independent lower, but also the new additional higher-order cell-dependent free vibration frequencies can be derived and analysed. The higher free vibration frequencies cannot be determined applying asymptotic models commonly used for***

investigations of dynamics of the periodic or tolerance-periodic shells. It has been shown that the differences between the fundamental lower free vibration frequencies derived from the tolerance model and free vibration frequencies obtained from the asymptotic one are ***negligibly small.*** Thus, *the effect of the microstructure size on the fundamental lower free vibration frequencies of the shells under consideration can be neglected.* Hence, ***the asymptotic model being more simple than the tolerance non-asymptotic one is sufficient from the point of view of calculations made for the vibration problems under consideration.***

The third aim of the dissertation has been to apply the microscopic equations (5.34)-(5.36) derived in the second step of the combined asymptotic-tolerance modelling to the analysis of length-scale effect in some special problems dealing with the shell micro-vibrations and with the long wave propagation related to micro-fluctuations of the shell displacements. These equations are independent of solutions obtained in the framework of the consistent asymptotic model (i.e. model derived in the first step of combined modelling) ***and make it possible to analyse selected problems of the shell micro-dynamics independently of the shell macro-dynamics. This is the greatest advantage of the proposed combined model.*** Moreover, micro-dynamic equations (5.34)-(5.36) involve terms with time and spatial derivatives of unknown micro-fluctuation amplitudes. Hence, they describe certain time-boundary layer and space-boundary layer phenomena strictly related to the specific form of initial and boundary conditions imposed on unknown fluctuation amplitudes.

It has been evidenced that using these micro-dynamic equations, ***the new cell-dependent higher free vibration frequencies can be determined and analysed <u>independently</u> of the fundamental, classical cell-independent lower free vibration frequencies.*** Since equations (5.34)-(5.36) contain continuously slowly-varying coefficients, the known Galerkin method was used to obtain approximate formulas of free micro-vibration frequencies.

Some new important results have been obtained analysing the harmonic micro-vibrations with vibration frequency $\breve{\omega}$. It has been shown that the form of these micro-vibrations depends on relations between values of vibration frequency $\breve{\omega}$ and a certain new additional higher-order free vibration frequency $\breve{\omega}_*$ $(7.19)_2$ depending on the cell size. ***The micro-vibrations can decay exponentially. They can decay linearly. For certain interrelations between*** $\breve{\omega}$ ***and*** $\breve{\omega}_*$ ***we deal with a non-decayed form of micro-vibrations (micro-vibrations oscillate) or with resonance micro-vibrations. Moreover, it has been shown that the micro-dynamic equations of the combined model describe the space-boundary layer***

phenomena.

Some new important results have been obtained analysing the long wave propagation problem related to micro-fluctuations in axial direction. It was shown that *the tolerance-periodic micro-heterogeneity of the shells leads to exponential waves and to dispersion effects.* Moreover, *the new wave propagation speed (7.35) depending on the microstructure size has been obtained.*

All the above length-scale problems studied within the micro-dynamic equations (5.34)-(5.36) of the combined model cannot be analysed in the framework of the asymptotic models commonly used for investigations of dynamic behaviour of the cylindrical shells with a functionally graded macrostructure and a tolerance-periodic microstructure.

The functionally graded shells being objects of considerations in this doctoral thesis are widely applied in civil engineering, most often as roof girders and bridge girders.

The results obtained in the dissertation generate new directions of further investigations. The anticipated directions of investigations can be related to: the modelling of stationary and dynamic stability problems in the framework of linear Kirchhoff-Love second-order theory, the non-linear shell dynamics and stability, the modelling of dynamic thermoelasticity problems and others.

Streszczenie

Dynamika cienkich powłok walcowych o funkcyjnej gradacji własności - modelowanie tolerancyjne

Tematem rozprawy doktorskiej jest matematyczne modelowanie zagadnień dynamiki cienkich, liniowo-sprężystych, mikro-niejednorodnych powłok walcowych typu Kirchhoffa-Love'a. ***W kierunku obwodowym, powłoki te w skali mikro mają tolerancyjnie periodyczną mikrostrukturę, natomiast w skali makro charakteryzują się funkcyjną poprzeczną gradacją własności uśrednionych (makrowłasności)***, por. rysunki 4.1, 4.2. Oznacza to, że na poziomie mikro, rozpatrywane w rozprawie powłoki zbudowane są z dużej liczby elementów (komórek) idealne ze sobą połączonych i regularnie rozmieszczonych w kierunku obwodowym. Zakłada się, że sąsiadujące ze sobą komórki są prawie identyczne, tzn. mają prawie identyczne własności geometryczne i materiałowe, natomiast komórki oddalone od siebie mogą znacznie się różnić. Zakłada się, że charakterystyczny wymiar liniowy komórki jest dostatecznie duży w porównaniu z maksymalną grubością powłoki oraz dostatecznie mały w porównaniu z minimalnym promieniem krzywizny oraz wymiarem liniowym powierzchni środkowej wzdłuż współrzędnej $x \equiv x^1$ parametryzującej tę powierzchnię w kierunku obwodowym. ***Na poziomie mikroskopowym***, własności geometryczne, sprężyste oraz inercyjne takich powłok opisane są silnie oscylującymi, nieciągłymi, tolerancyjnie periodycznymi funkcjami podług argumentu x. Natomiast, ***na poziomie makroskopowym***, uśrednione własności rozpatrywanych powłok są opisane funkcjami ciągłymi i wolnozmiennymi w kierunku tolerancyjnej periodyczności. Ponadto, uśrednione własności zmieniają się w kierunku prostopadłym do granic między składnikami. Powłoki takie są nazywane powłokami ***o funkcyjnej poprzecznej gradacji makrowłasności.***

W kierunku osiowym, własności rozpatrywanych powłok są stałe.

Opis dynamicznych zachowań mikro-niejednorodnych powłok będących przedmiotem rozprawy, w ramach znanej teorii Kirchhoffa-Love'a, prowadzi do równań, których współczynniki są tolerancyjnie periodycznymi, silnie oscylującymi i nieciągłymi funkcjami w kierunku obwodowym. Stąd, równania te nie mogą być

wprost zastosowane do analizy zagadnień inżynierskich. Formułowane są zatem różne przybliżone metody modelowania (tj. procedury uśredniające) prowadzące od równań różniczkowych cząstkowych z silnie oscylującymi współczynnikami do równań o współczynnikach ciągłych i wolnozmiennych (lub współczynnikach stałych w przypadku walcowych powłok periodycznych).

Modele uśrednione powłok (płyt) periodycznych/tolerancyjnie periodycznych są najczęściej otrzymywane na drodze ***homogenizacji asymptotycznej***. Jednakże, modele te pomijają wpływ wielkości komórki na globalne (makroskopowe) zachowania powłoki, tzn. pomijają efekt skali.

Alternatywne, nieasymptotyczne podejście do matematycznego modelowania ciał periodycznych lub tolerancyjnie periodycznych, oparte na ***pojęciu tolerancji*** (pojęcie związane z dokładnością prowadzonych pomiarów lub obliczeń) i prowadzące do uśrednionych równań o stałych lub ciągłych i wolnozmiennych współczynnikach zależnych od wielkości komórki, zostało zaproponowane i rozwijane przez profesora Czesława Woźniaka w wielu publikacjach i podsumowane w monografiach [164, 166, 168]. ***Relacje tolerancyjne determinowane parametrami tolerancji, funkcje wolno-zmienne, funkcje tolerancyjno-periodyczne, fluktuacyjne funkcje kształtu oraz operacja uśredniania są podstawowymi pojęciami techniki tolerancyjnego modelowania***. Technika ta oparta jest na dwóch założeniach. Pierwsze z tych założeń, zwane ***przybliżeniem (uśrednieniem) tolerancyjnym***, umożliwia pomijanie wyrazów rzędu parametrów tolerancji. Drugie założenie zwane jest ***mikro-makro dekompozycją*** pól przemieszczeń (lub pola temperatury w zagadnieniach przepływu ciepła). Ograniczając się do zagadnień mechanicznych, zgodnie z tym założeniem ***nieznane przemieszczenia*** w równaniach wyjściowych są przedstawione w postaci sumy ***nieznanych uśrednionych na komórce przemieszczeń***, będących ***funkcjami wolnozmiennymi*** (tzn. przyjmującymi w ramach tolerancji stałe wartości na komórce), oraz ***silnie oscylujących fluktuacji***. Fluktuacje są opisane ***przez znane w każdym analizowanym zagadnieniu, liniowo-niezależne, periodyczne lub tolerancyjnie periodyczne fluktuacyjne funkcje kształtu pomnożone przez nieznane wolnozmienne funkcje, zwane amplitudami fluktuacji.***

W niniejszej rozprawie wykorzystano ***technikę tolerancyjnego modelowania*** w zagadnieniach dynamiki walcowych powłok mikro-niejednorodnych o poprzecznej gradacji makrowłasności. Wykorzystano także nowe podejście do asymptotycznego uśredniania równań różniczkowych cząstkowych (lub funkcjonałów całkowych) o silnie oscylujących periodycznych/tolerancyjnie periodycznych współczynnikach przedstawione w książce [164] pod redakcją Cz. Woźniaka. Podejście to nazwano asymptotycznym konsystentnym. Wykorzystano również zaproponowaną w monografii [164] połączoną asymptotyczno-tolerancyjną technikę modelowania ciał mikro-niejednorodnych. Stosując modelowanie tolerancyjne, asymptotyczne konsystentne oraz asymptotyczno-tolerancyjne do wyjściowych równań Eulera-Lagrange'a (4.9), których jawna postać pokrywa się ze znanymi równaniami (4.10) teorii Kirchhoffa-Love'a cienkich powłok sprężystych, ***wyprowadzono trzy nowe matematyczne uśrednione modele: tolerancyjny, asymptotyczny konsystentny oraz asymptotyczno-tolerancyjny***. W przeciwieństwie do silnie oscylujących i nieciągłych współczynników równań wyjściowych, współczynniki równań różniczkowych tych uśrednionych modeli są ciągłymi i wolnozmiennymi funkcjami podług współrzędnej x parametryzującej powierzchnię środkową powłoki w kierunku obwodowym. Ponadto, modele tolerancyjny i asymptotyczno-tolerancyjny uwzględniają wpływ wielkości mikrostruktury na dynamiczne zachowania powłoki. Wpływ ten zwany jest ***efektem skali***.

Procedura tolerancyjnego modelowania zastosowana do wyjściowych równań Eulera-Lagrange'a (4.9) realizowana była w dwóch etapach. Pierwszy etap polegał na ***tolerancyjnym uśrednieniu funkcji Lagrange'a*** (4.8) z wykorzystaniem mikro-makro dekompozycji (5.1), operacji uśredniania (3.5) oraz przybliżenia tolerancyjnego (3.6). Tolerancyjnie uśredniona postać lagrangianu (4.8) dana jest wzorem (5.3). W drugim etapie, stosując ***zasadę stacjonarności działania do uśrednionego funkcjonału działania*** (5.4) zdefiniowanego poprzez ***tolerancyjny uśredniony lagrangian*** (5.3), otrzymano uśrednione równania Eulera-Lagrange'a (5.5), których jawna postać reprezentowana jest przez relacje konstytutywne (5.6) oraz równania ruchu (5.7). Równania (5.6), (5.7) wraz z mikro-makro dekompozycją (5.1) reprezentują model tolerancyjny do analizy zagadnień dynamiki cienkich powłok walcowych o tolerancyjnie periodycznej mikrostrukturze oraz o funkcyjnej poprzecznej gradacji makrowłasności w kierunku obwodowym. Współczynniki równań modelu tolerancyjnego są ciągłe i wolnozmienne. Ponadto, niektóre z tych współczynników zależą od wielkości mikrostruktury. Efekt skali może być analizowany nie tylko w dynamicznych, ale także w stacjonarnych zagadnieniach. Niewiadome równań modelu, tzn. ***makroprzemieszczenia oraz amplitudy fluktuacji, muszą być funkcjami wolnozmiennymi w kierunku tolerancyjnej periodyczności***. Te wymagania są wykorzystane do oceny *a posteriori* parametrów tolerancji, czyli także do sprawdzenia fizycznej poprawności wyników otrzymanych w ramach modelu tolerancyjnego.

Technika ***asymptotycznego konsystentnego uśredniania*** równań różniczkowych cząstkowych (lub funkcjonałów całkowych) o silnie oscylujących periodycznych/tolerancyjnie periodycznych współczynnikach, przedstawiona w monografii [164], nie zawiera pojęć funkcji tolerancyjnie periodycznej i funkcji wolnozmiennej. Wprowadzono tu tylko pojęcie fluktuacyjnej funkcji kształtu. W zagadnieniach mechanicznych, podstawowym kinematycznym założeniem tego podejścia jest ***asymptotyczna dekompozycja pól przemieszczeń***. Zgodnie z tym założeniem, przemieszczenia występujące w wyjściowych równaniach (lub w wyjściowym funkcjonale całkowym) zastąpione są rodzinami pól przemieszczeń zdefiniowanymi na komórce i zależnymi od parametru $\varepsilon \in (0, 1]$. Rodziny te rozłożone są na ***nieznane przemieszczenia***, (zwane tak jak w podejściu tolerancyjnym ***makroprzemieszczeniami***), ***niezależne od parametru*** ε ***oraz silnie oscylujące fluktuacje przemieszczeń zależne od*** ε. Te silnie oscylujące fluktuacje są reprezentowane przez *znane periodyczne/tolerancyjnie periodyczne* ***fluktuacyjne funkcje kształtu zależne od*** ε oraz przez ***nieznane funkcje niezależne od*** ε, które, tak jak w podejściu tolerancyjnym, zwane są ***amplitudami fluktuacji***. Żąda się, aby występujące w asymptotycznej dekompozycji funkcje niezależne od ε były ciągłe i ograniczone w kierunkach periodyki lub tolerancyjnej periodyczności wraz z ich odpowiednimi pochodnymi. Niezależność wyżej wymienionych funkcji od parametru ε stanowi zasadniczą różnicę między podejściem asymptotycznym konsystentnym a podejściem stosowanym w znanych teoriach homogenizacji asymptotycznej. Ponadto, modele asymptotyczne konsystentne, w przeciwieństwie do powszechnie stosowanych modeli asymptotycznych, nie wymagają rozwiązywania skomplikowanych analitycznie brzegowych zagadnień na komórce w celu wyznaczenia efektywnych sztywności ciała. W podejściu asymptotycznym konsystentnym, moduły efektywne są rozwiązaniem układu równań algebraicznych liniowych dla nieznanych *amplitud fluktuacji*.

Procedura modelowania asymptotycznego konsystentnego zastosowana do wyjściowych równań Eulera-Lagrange'a (4.9) realizowana była w dwóch etapach. Pierwszy etap polegał na ***asymptotycznym konsystentnym uśrednieniu funkcji Lagrange'a*** (4.8) z wykorzystaniem asymptotycznej dekompozycji (5.8). Uśredniona postać lagrangianu (4.9) dana jest wzorem (5.12). W drugim etapie, stosując ***zasadę stacjonarności działania do uśrednionego funkcjonału działania*** (5.13) zdefiniowanego poprzez ***asymptotycznie uśredniony lagrangian*** (5.12), otrzymano uśrednione równania Eulera-Lagrange'a (5.14) oraz ich jawną postać (5.15). Po wyeliminowaniu z układu równań (5.15) amplitud fluktuacji, por. wzory (5.16), otrzymano równania modelu asymptotycznego (5.18) wyrażone tylko w makroprzemieszczeniach. Równania (5.18) wraz z dekompozycją (5.19) oraz równaniami dla amplitud fluktuacji (5.16) reprezentują model asymptotyczny konsystentny do analizy zagadnień dynamiki rozważanych w rozprawie cienkich powłok walcowych o tolerancyjnie periodycznej mikrostrukturze oraz o funkcyjnej poprzecznej gradacji makrowłasności w kierunku obwodowym. Współczynniki równań modelu asymptotycznego są ciągłe i wolno zmieniające się. ***Współczynniki te nie zależą od parametru długości***

mikrostruktury.

Stosując procedurę asymptotyczno-tolerancyjną, por. książka [164], wyprowadzono model asymptotyczno-tolerancyjny do badania zagadnień dynamiki walcowych powłok o tolerancyjnie periodycznej mikrostrukturze i poprzecznej gradacji makrowłasności w kierunku obwodowym. Równania tego modelu reprezentowane są przez równania (5.20) modelu asymptotycznego (makroskopowego), sformułowane z zastosowaniem techniki modelowania asymptotycznego konsystentnego i mające ciągłe, wolno zmieniające się współczynniki ***niezależne od wielkości mikrostruktury*** oraz równania (5.30), (5.31) modelu tolerancyjnego (mikroskopowego), wyprowadzone z zastosowaniem techniki tolerancyjnego modelowania i mające ciągłe, wolnozmienne współczynniki ***zależne od wielkości komórki***. Obydwa modele połączone są ze sobą na podstawie założenia, że rozwiązania danego zagadnienia brzegowo-początkowego w ramach modelu asymptotycznego są znane. W rozprawie pokazano, że przy pewnych warunkach nałożonych na fluktuacyjne funkcje kształtu, otrzymuje się równania (5.34)-(5.36) niezależne od rozwiązań w ramach modelu asymptotycznego. ***Równania te opisują mikrodynamiczne zachowania rozważanych w rozprawie powłok niezależnie od ich makrodynamicznych zachowań.*** Jest to główna zaleta wyprowadzonego modelu asymptotyczno-tolerancyjnego.

Sformułowane modele tolerancyjny i asymptotyczny zastosowano do ***oceny efektu skali*** w pewnych szczególnych zagadnieniach dotyczących drgań własnych rozważanych powłok. Ponieważ znalezienie analitycznych rozwiązań równań modelu tolerancyjnego lub modelu asymptotycznego w ogólnym przypadku nie jest możliwe, zastosowano przybliżony sposób rozwiązania. Przybliżone wzory częstości drgań własnych otrzymano korzystając z metody Ritza. ***Analizując dynamikę powłok w ramach modelu tolerancyjnego, otrzymano wzory analityczne nie tylko na podstawowe tzw. niższe częstości drgań własnych, ale również na nowe, dodatkowe tzw. wyższe częstości drgań własnych zależne od parametru długości mikrostruktury. Wyższe częstości umożliwiają analizę drgań wyższego rzędu oraz zjawiska dyspersji. Te nowe wyższe częstości drgań nie mają swoich odpowiedników w modelach asymptotycznych oraz w modelach numerycznych, opartych na przykład na metodzie elementów skończonych.*** Wykazano, że wartości niższych częstości drgań własnych obliczane według modelu tolerancyjnego są pomijalnie większe od wartości odpowiednich częstości drgań otrzymanych w ramach modelu asymptotycznego. Uwzględnienie efektu skali powoduje więc jedynie nieznaczną, nie mającą praktycznego znaczenia korektę wartości częstości drgań. Oznacza to, że ***efekt skali w zagadnieniach dotyczących drgań własnych rozważanych powłok jest pomijalnie mały z obliczeniowego***

punktu widzenia. Zagadnienia te mogą być analizowane w ramach modeli asymptotycznych (prostszych analitycznie od modeli z efektem skali).

Wyprowadzone z zastosowaniem procedury asymptotyczno-tolerancyjnej równania (5.34)-(5.36), niezależne od rozwiązań w ramach modelu asymptotycznego, wykorzystano do analizy zagadnień mikrodynamiki rozważanych tolerancyjnie periodycznych powłok.

Analizowano mikrodrgania niezależnie od makrodrgań. Wyprowadzono wzory na ***wyższe, dodatkowe, zależne od wielkości mikrostruktury częstości mikrodrgań własnych w kierunkach obwodowym, osiowym oraz normalnym do powierzchni środkowej***. Wzory te otrzymano korzystając z przybliżonej metody Galerkina. ***Przeprowadzono dokładną analizę tych częstości***.

Analizując harmoniczne mikrodrgania w kierunku osiowym z częstością $\breve{\omega}$ (rozprzężone z mikrodrganiami w kierunkach obwodowym i normalnym), uzyskano nowe wyniki w teorii mikrodrgań powłok o funkcyjnej gradacji własności. Pokazano, że w zależności od relacji między częstością drgań harmonicznych $\breve{\omega}$ a wyższą częstością mikrodrgań własnych $\breve{\omega}_*$, zależną od wielkości komórki, występują różne postaci mikrodrgań. ***Mikrodrgania zanikają wykładniczo lub liniowo. Dla pewnych relacji między wartościami $\breve{\omega}$ i $\breve{\omega}_*$ mamy do czynienia z niezanikającą (tj. oscylującą) postacią mikrodrgań lub z mikrodrganiami rezonansowymi. Zbadano także tzw. efekt warstwy brzegowej, gdzie termin „brzeg" odnosił się do przestrzeni.***

Nowe wyniki uzyskano badając zagadnienia propagacji fal długich w nieograniczonych w kierunku osiowym powłokach tolerancyjnie periodycznych. Badane fale odnosiły się tylko do fluktuacyjnych części przemieszczeń zależnych od efektu skali. ***Pokazano, że w zależności od ograniczeń nałożonych na prędkość propagacji fal mogą propagować się trzy typy fal***: sinusoidalna lub wykładnicza lub występuje zdegenerowany przypadek rozgraniczający fale sinusoidalne i wykładnicze. Wyprowadzono relacje dyspersji. ***Wyprowadzono i zbadano nową prędkość propagacji fal zależną od parametru długości mikrostruktury.***

Nowe wyniki uzyskano analizując szczególny problem początkowy opisany równaniem (5.35) dla amplitud mikro-fluktuacji w kierunku osiowym. W badanym zagadnieniu równanie to redukuje się do równania różniczkowego zwyczajnego drugiego rzędu z pochodnymi względem czasu. Problem ten ilustruje wpływ wielkości komórki na charakter mikro-fluktuacji przemieszczeń w kierunku osiowym, przy przyjętych warunkach początkowych. ***Pokazano, że w zależności od relacji między pewnym parametrem długości l niezależnym od wielkości komórki a parametrem długości mikrostruktury λ,***

mikro-fluktuacje przemieszczeń w kierunku osiowym mają różny charakter w badanym przedziale czasu: dla $l < \lambda$ maleją monotonicznie i bardzo łagodnie i nie przyjmują wartości zero w badanym przedziale czasu, dla $l = \lambda$ mikro-fluktuacje maleją monotonicznie i na końcu badanego przedziału czasu są równe zeru, dla $l > \lambda$ mikro-fluktuacje zanikają monotonicznie i silnie w pewnym podprzedziale badanego przedziału czasu, a następnie absolutne wartości mikro-fluktuacji rosną monotonicznie w pozostałej części tego przedziału czasu.

Wszystkie przedstawione powyżej efekty, uzyskane w ramach równań mikromechaniki (5.34)-(5.36) modelu asymptotyczno-tolerancyjnego ***nie mogą być analizowane w ramach asymptotycznych modeli powłok, jak również przy użyciu znanych programów komputerowych.***

Wyprowadzone w niniejszej rozprawie modele mikro-niejednorodnych powłok walcowych mogą być wykorzystane do badań dynamiki powłokowych elementów konstrukcyjnych mostów i dachów, powłokowych elementów reaktorów, powłokowych elementów samolotów, okrętów, maszyn.

Uzyskane wyniki mają istotny wpływ na stan wiedzy dotyczącej dynamicznych zachowań cienkościennych powłok walcowych o ***tolerancyjnie periodycznej mikrostrukturze oraz o funkcyjnej poprzecznej gradacji własności uśrednionych (makrowłasności) w kierunku obwodowym***, a także generują nowe kierunki badań i tym samym wywierają wpływ na rozwój tej dziedziny wiedzy.

Dalsze badania mikro-niejednorodnych powłok walcowych, będących obiektem rozważań rozprawy doktorskiej, z wykorzystaniem techniki tolerancyjnego modelowania mogą dotyczyć nieliniowych zagadnień dynamiki i stateczności, problemów dynamicznej termosprężystości, formułowania modeli w ramach teorii dokładniejszych niż teoria Kirchhoffa-Love'a.